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On moment maps associated to a twisted Heisenberg double

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Abstract

We review the concept of the (anomalous) Poisson-Lie symmetry in a way that emphasises the notion of Poisson-Lie Hamiltonian. The language that we develop turns out to be very useful for several applications: we prove that the left and the right actions of a group G on its twisted Heisenberg double (D, κ) realize the (anomalous) Poisson-Lie symmetries and we explain in a very transparent way the concept of the Poisson-Lie subsymmetry and that of Poisson-Lie symplectic reduction. Under some additional conditions, we construct also a non-anomalous moment map corresponding to a sort of quasi-adjoint action of G on (D, κ) . The absence of the anomaly of this "quasi-adjoint" moment map permits to perform the gauging of deformed WZW models.

1 Introduction

Poisson-Lie symmetry [15] is the generalization of the ordinary Hamiltonian symmetry of a dynamical system and, upon quantizing, it becomes the quantum group symmetry. Many dynamical systems can be deformed in such a way that their ordinary symmetries become Poisson-Lie. Among such systems there is also the standard WZW model [17] where the loop group symmetry gets deformed [9]. The principal goal of the present work is to develop the theory of gauging of the deformed WZW model.

From the mathematical point of view, the problem amounts to identify non-anomalous Poisson-Lie subsymmetries of the deformed WZW model which would permit to perform the gauging. In order to describe the Poisson-Lie analogue of the WZW vanishing anomaly condition [18], first we shall have to develop appropriate mathematical tools. In particular, it turns out that the standard definition of the Poisson-Lie symmetry (i.e. the action map $G \times M \rightarrow M$ is Poisson) is too rough since it is unable to distinguish between non-anomalous and anomalous symmetries. For this reason, we shall refine the standard concept of the Poisson-Lie symmetry and propose its new definition based rather on the Poisson-Lie structure on the cosymmetry (or dual) group B than on the symmetry group G . We are fully aware that the language that we develop is not quite standard in the Poisson-(Lie) geometry but we find it well adapted for our discussion of anomalies and we also believe that it may constitute an insightful alternative in treating the Poisson-Lie symmetric systems in general.

The central object of our investigations will be a class of Poisson manifolds introduced by Semenov-Tian-Shansky under the name of twisted Heisenberg doubles [16]. As it was conjectured in [9] and showed in [11], particular elements of this class play the role of the phase spaces of the deformed WZW models. This also means that results obtained in full generality for any twisted Heisenberg double will also hold for any deformed WZW model.

In order to present in this introduction the principal ideas and results of our work, we first expose two main definitions and three main theorems proved later in the body of the paper.

Definition 1: Let M be a symplectic manifold whose algebra of smooth functions $Fun(M)$ is equipped with a Poisson bracket $\{.,.\}$. Let B be a

Poisson-Lie group and let $\mu : M \rightarrow B$ be a smooth map. To every function $y \in Fun(B)$ we can associate a vector field $w_\mu(y) \in Vect(M)$ as follows:

$$w_\mu(y)f = \{f, \mu^*(y')\} \mu^*(S(y'')), \quad y \in Fun(B), f \in Fun(M).$$

We say that μ realizes the Poisson-Lie symmetry of M if the map w_μ is homomorphism of the Lie algebras $Fun(B)$ and $Vect(M)$. If, moreover, the map μ is Poisson, we say that the symmetry is equivariant or non-anomalous.

Definition 2: Let D be an even-dimensional Lie group equipped with a maximally Lorentzian bi-invariant metric. If $Lie(D) = Lie(G) \dot{+} Lie(B)$, where G and B are maximally isotropic subgroups, D is called the Drinfeld double of G or the Drinfeld double of B . Let κ be a metric preserving automorphism of D and suppose that there are respective basis T^i and t_i ($i = 1, \dots, n$) of $\mathcal{G} = Lie(G)$ and $\mathcal{B} = Lie(B)$ such that

$$(T^i, t_j)_D = \delta_j^i.$$

Then the (basis independent) expression

$$\{f_1, f_2\}_D \equiv \nabla_{T^i}^R f_1 \nabla_{t_i}^R f_2 - \nabla_{\kappa(t_i)}^L f_1 \nabla_{\kappa(T^i)}^L f_2, \quad f_1, f_2 \in Fun(D)$$

is a Poisson bracket and the Poisson manifold $(D, \{.,.\}_D)$ is called the twisted Heisenberg double.

Theorem 1: Let D be a twisted Heisenberg double which is also decomposable, i.e. such that two global unambiguous decompositions hold: $D = \kappa(B)G$ and $D = \kappa(G)B$. Consider (smooth) maps $\Lambda_L, \Lambda_R : D \rightarrow B$, $\Xi_R, \Xi_L : D \rightarrow G$ respectively induced by these two decompositions. Then it holds:

- a) The Poisson manifold $(D, \{.,.\}_D)$ is symplectic.
- b) Both maps Λ_L and Λ_R realize the (anomalous) Poisson-Lie symmetries of the symplectic manifold $(D, \{.,.\}_D)$. The corresponding symmetry group is G acting as

$$h \triangleright K = \kappa(h)K, \quad h \in G, \quad K \in D$$

or, respectively, as

$$h \triangleright K = Kh^{-1}, \quad h \in G, \quad K \in D.$$

Theorem 2: Let D be a decomposable twisted Heisenberg double such that the twisting automorphism κ preserves the subgroup B . Construct two new maps $B_L : D \rightarrow B$ and $B_R : D \rightarrow B$ as follows

$$B_L(K) = \kappa(\Lambda_L(K))\Lambda_R(K), \quad B_R(K) = \kappa^{-1}(\Lambda_R(K))\Lambda_L(K), \quad K \in D.$$

Then it holds: Both maps B_L and B_R are Poisson and they realize the (non-anomalous) Poisson-Lie symmetries of $(D, \{.,.\}_D)$. The corresponding symmetry group is G acting as

$$h \triangleright K = \kappa(h)K\Xi_R(\kappa[h\Lambda_L(K)]), \quad h \in G, \quad K \in D,$$

or, respectively, as

$$h \triangleright K = \kappa[\Xi_L^{-1}(\Lambda_R^{-1}(K)h^{-1})]Kh^{-1}. \quad h \in G, \quad K \in D.$$

Theorem 3: Let D be a decomposable twisted Heisenberg double, κ an automorphism of D preserving B and N a normal subgroup of B . Denote by C the factor group B/N , by ρ the natural homomorphism $B \rightarrow C$ and by $P_\kappa : Lie(D) \rightarrow Lie(B)$ a projector on $Lie(B)$ with kernel $\kappa(Lie(G))$. Suppose that the Hopf subalgebra $\rho^*(Fun(C))$ of $Fun(B)$ is also a Poisson subalgebra. Then it holds: The composed map $\nu_R \equiv \rho \circ \Lambda_R$ realizes the Poisson-Lie symmetry of D and the corresponding symmetry group H is the subgroup of G . If, moreover, $P_\kappa(Lie(H)) \subset Lie(N)$ then the moment map ν_R is non-anomalous.

Apart from these three theorems, we prove two more propositions (Lemma 3 and Lemma 4) enlarging the story to the non-decomposable twisted Heisenberg doubles. The formulations of those additional Lemmas require introduction of several new concepts therefore, for the sake of conciseness of this introduction, we shall expose them only in Section 3.3.

The principal field of applications of our results is the theory of non-linear σ -models which are two-dimensional field theories describing the propagation of closed strings on a Riemannian manifold T . The manifold T is often referred to as the target space and it comes also equipped with a closed 3-form H . The classical action for a closed string configuration $x^\mu(\sigma, \tau)$ reads

$$S[x^\mu(\sigma, \tau)] = \frac{1}{2} \int d\sigma d\tau G_{\mu\nu}(x) \partial_+ x^\mu \partial_- x^\nu + \int_V x^* H,$$

where σ is a periodic loop parameter, τ the evolution parameter, x^μ are coordinates on T , $G_{\mu\nu}$ are the components of the Riemannian metric and

$$\partial_\pm \equiv \partial_\tau \pm \partial_\sigma.$$

It should be noted that the configuration $x^\mu(\sigma, \tau)$ is extended to a configuration defined in the volume V whose boundary is the surface of the propagating closed string and x^*H is the pull-back of the H -potential to this volume V . A detailed explanation why the variational principle based on the action S does not depend on the ambiguity of the extension of x is given e.g. in [17, 6, 12]. The prominent example of the non-linear σ -model is the WZW model for which the target space is the compact group manifold K equipped with the standard Killing-Cartan metric $(\cdot, \cdot)_K$. Its action reads

$$S_{WZW}[g(\sigma, \tau)] = \frac{1}{2} \int d\sigma d\tau (\partial_+ g g^{-1}, \partial_- g g^{-1})_K + \frac{1}{12} \int_V ([d g g^{-1}, d g g^{-1}], d g g^{-1})_K.$$

Let S be a subgroup of K and let $A_\pm(\sigma, \tau)$ be two $Lie(S)$ -valued fields. The gauged K/S WZW model is then a dynamical system described by the following classical action

$$S_{GWZW}[g(\sigma, \tau), A_\pm(\sigma, \tau)] = S_{WZW}[g(\sigma, \tau)] + \int d\sigma d\tau \left(-(\partial_+ g g^{-1}, A_-)_K + (\partial_- g g^{-1}, A_+)_K - (g^{-1} A_- g, A_+)_K + (A_-, A_+)_K \right).$$

The action S_{GWZW} is invariant with respect to gauge transformations

$$g(\sigma, \tau) \rightarrow s^{-1}(\sigma, \tau) g(\sigma, \tau) s(\sigma, \tau),$$

$$A_\pm(\sigma, \tau) \rightarrow s^{-1}(\sigma, \tau) A_\pm(\sigma, \tau) s(\sigma, \tau) - s^{-1}(\sigma, \tau) \partial_\pm s(\sigma, \tau),$$

where $s(\sigma, \tau)$ takes values in the subgroup S .

(Gauged) WZW models are dynamical systems whose phase spaces are symplectic manifolds. We shall show in Section 4, that their symplectic structures coincide with those of (gauged) twisted Heisenberg doubles. Actually, the twisted Heisenberg doubles underlying the ordinary WZW models are very special in the sense that the symmetry group G is the loop group LK and the cosymmetry group B is Abelian. If we consider also doubles with non-Abelian B , we are very naturally led to more general theories which we

call the deformed WZW models. Let us now explain the meaning of the Theorems 1,2 and 3 in the WZW context.

If B is Abelian, the Theorem 1 says that the ordinary WZW models enjoy two anomalous chiral symmetries respectively given by the (twisted) left and ordinary right multiplications by elements of the loop group LK . If B is non-Abelian, the deformed WZW models still have two anomalous chiral Poisson-Lie symmetries. Theorem 2 says that the left and right moment maps Λ_L, Λ_R can be combined into the non-anomalous moment maps B_L, B_R . For B Abelian, this new moment maps are equal to each other and they generate the adjoint action of G on the target space of the σ - model. This adjoint action is non-anomalous and serves as the base of the standard vector gauging of the WZW model leading to the gauged K/S WZW model described above. However, if B is non-Abelian, the moment maps B_L and B_R do not coincide and we have two different non-anomalous quasi-adjoint actions of Theorem 2 which can be consistently gauged. Finally, the Theorem 3 explains under which conditions the chiral subsymmetries may become non-anomalous and can be consistently gauged. As an illustration, we devote an entire Section 4 to a very explicite construction of a particular new deformation of the ordinary WZW model (which we call the u -deformation) and work out in detail its deformed vector gauging.

The paper is organized as follows: In Section 2, we present the discussion of the concept of the Poisson-Lie symmetry, we explain motivations for the Definition 1 and we prove the Theorem 1. Then in Section 3.1 and 3.2, we respectively prove the Theorems 2 and 3 and, in Section 3.3, we expose the theory of the non-decomposable doubles. In the section 4, we construct the u -deformed WZW model and perform its Poisson-Lie gauging. We finish with short conclusions and an outlook.

2 Twisted Heisenberg double

The presentation of this Section extends that of [11]. In particular, we give full proofs of the statements listed in [11], and, moreover, we are more general concerning the properties of the twist κ of a double D .

2.1 Lie groups in a dual language

Let B be a Lie group and $Fun(B)$ the algebra of functions on it. It is well known that the group structure on B gives rise to a so called coproduct $\Delta : Fun(B) \rightarrow Fun(B) \otimes Fun(B)$, the antipode $S : Fun(B) \rightarrow Fun(B)$ and the counit $\varepsilon : Fun(B) \rightarrow \mathbf{R}$ given, respectively, by the formulae

$$\Delta x(b_1, b_2) = x'(b_1)x''(b_2) = x(b_1b_2), \quad S(x)(b) = x(b^{-1}), \quad \varepsilon(x) = x(e_B).$$

Here $x \in Fun(B)$, $b, b_1, b_2 \in B$, e_B is the unit element of B and we use the Sweedler notation for the coproduct:

$$\Delta x = \sum_{\alpha} x'_{\alpha} \otimes x''_{\alpha} \equiv x' \otimes x''.$$

The Lie algebra \mathcal{B} of B is defined as the set of ε -derivations of $Fun(B)$, i.e.

$$\mathcal{B} = \{\delta : Fun(B) \rightarrow \mathbf{R}, \delta(xy) = \varepsilon(x)\delta(y) + \varepsilon(y)\delta(x)\}.$$

The Lie bracket on \mathcal{B} is defined as follows:

$$[\delta_1, \delta_2](x) = \delta_1(x')\delta_2(x'') - \delta_1(x'')\delta_2(x').$$

This definition of the Lie algebra \mathcal{B} is of course equivalent to a more standard one presenting \mathcal{B} as the set of right-invariant vector fields. In order to connect two definitions, consider a map $\phi^B : Fun(B) \rightarrow \Omega^1(B)$ (the map ϕ^B thus goes from functions into 1-forms on B) defined by

$$\phi^B(x) = dx'S(x'').$$

Note that the 1-form $\phi^B(x)$ is automatically right-invariant therefore the canonical pairing of a right-invariant vector field v with $\phi^B(x)$ defines a map $\delta_v : Fun(B) \rightarrow \mathbf{R}$:

$$\delta_v(x) = \langle v, \phi^B(x) \rangle. \quad (0)$$

The map δ_v is indeed the ε -derivation due to the following property of the map ϕ^B :

$$\phi^B(xy) = \varepsilon(x)\phi^B(y) + \varepsilon(y)\phi^B(x).$$

On the other hand, every ε -derivation δ defines a right-invariant vector field ∇_{δ}^L which acts on $x \in Fun(B)$ as follows:

$$\nabla_{\delta}^L x = \delta(x')x''.$$

Consider now a Poisson-Lie group B , i.e. a Lie group equipped with a Poisson bracket $\{.,.\}_B$ satisfying

$$\Delta\{x, y\}_B = \{x', y'\}_B \otimes x''y'' + x'y' \otimes \{x'', y''\}_B, \quad x, y \in Fun(B). \quad (1)$$

It is not difficult to prove that the property (1) implies

$$S(\{x, y\}_B) = -\{S(x), S(y)\}_B, \quad x, y \in Fun(B) \quad (2a)$$

$$\varepsilon(\{x, y\}_B) = 0, \quad x, y \in Fun(B). \quad (2b)$$

Denote by \mathcal{B}^* the linear dual of the Lie algebra $\mathcal{B} = Lie(B)$. The Poisson-Lie bracket $\{.,.\}_B$ induces a natural Lie algebra structure $[\cdot, \cdot]^*$ on \mathcal{B}^* . Let us explain this fact in more detail: First of all recall that \mathcal{B}^* can be identified with the space of right-invariant 1-forms on the group manifold B and we have the natural (surjective) map $\phi^B : Fun(B) \rightarrow \mathcal{B}^*$ defined by

$$\phi^B(y) = dy'S(y''), \quad y \in Fun(B).$$

Note that the 1-form $\phi^B(y)$ is right-invariant therefore it is indeed in \mathcal{B}^* . Let $U, V \in \mathcal{B}^*$ and $x, y \in Fun(B)$ such that $U = \phi^B(x)$ and $V = \phi^B(y)$. Then we define

$$[U, V]^* = \phi^B(\{x, y\}_B). \quad (2c)$$

It is the Poisson-Lie property (1) of $\{.,.\}_B$ which ensures the independence of $[U, V]^*$ on the choice of the representatives x, y . In what follows, the Lie algebra $(\mathcal{B}^*, [\cdot, \cdot]^*)$ will be denoted by the symbol \mathcal{G} and G will be a (connected simply connected) Lie group such that $\mathcal{G} = Lie(G)$. We note that G is often referred to as the dual group of B . It can be itself equipped with a Poisson-Lie bracket $\{.,.\}_G$ inducing on $\mathcal{G}^* \equiv \mathcal{B}$ the correct Lie algebra structure $Lie(B)$.

2.2 Poisson-Lie symmetry

The concept of the Poisson-Lie symmetry of a symplectic manifold M was introduced by Semenov-Tian-Shansky [15]. Traditionally, it concerns the action of a Poisson-Lie group G on M such that the smooth map $G \times M \rightarrow M$ is Poisson. Certain Poisson-Lie symmetries have moment maps $\mu : M \rightarrow B$, where B is the dual Poisson-Lie group. Let Π_M be the Poisson bivector

corresponding to the symplectic structure on M , let ρ_B be the right-invariant Maurer-Cartan form on B and let $\langle \cdot, \cdot \rangle$ denote the canonical pairing between $Lie(B)$ and $Lie(G)$. Then the moment map μ is characterized by the property that the vector field $\Pi_M(\cdot, \mu^* \langle \rho_B, U \rangle) \in Vect(M)$ generates the infinitesimal action of the element $U \in Lie(G)$ on M . We have the following lemma:

Lemma 1 : Let the action $G \times M \rightarrow M$ be the Poisson-Lie symmetry with the moment map $\mu : M \rightarrow B$ and let $w_\mu : Fun(B) \rightarrow Vect(M)$ be a map defined as

$$w_\mu(y) = \Pi_M(\cdot, \mu^* \phi^B(y)).$$

Then w_μ is anti-homomorphism of the Lie algebras $Fun(B)$ and $Vect(M)$.

Proof: Let x, y be in $Fun(B)$. We know that the right-invariant 1-forms $\phi^B(x)$ and $\phi^B(y)$ can be seen as the elements of $Lie(G)$, denote them as U and V , respectively. Then the statement of the Lemma follows from Eq. (2c) and from the property of the moment map stated above.

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In this paper, we shall advocate a different approach to Poisson-Lie symmetry and we take the statement of the Lemma 1 as a definition. Thus we propose

Definition 1: Let M be a symplectic manifold whose algebra of smooth functions $Fun(M)$ is equipped with a Poisson bracket $\{\cdot, \cdot\}$. Let B be a Poisson-Lie group and let $\mu : M \rightarrow B$ be a smooth map. To every function $y \in Fun(B)$ we can associate a vector field $w_\mu(y) \in Vect(M)$ as follows:

$$w_\mu(y)f = \{f, \mu^*(y')\} \mu^*(S(y'')), \quad y \in Fun(B), f \in Fun(M). \quad (3)$$

We say that μ realizes the Poisson-Lie symmetry of M if the map w_μ is an anti-homomorphism of the Lie algebras $Fun(B)$ and $Vect(M)$. If, moreover, the map μ is Poisson, we say that the symmetry is equivariant or non-anomalous.

#

Explanations: If μ realizes the Poisson-Lie symmetry of M , the opposite Lie algebra of the image $Im(w_\mu)$ of the map w_μ is a Lie algebra that will be denoted as \mathcal{G} . If the action of the Lie algebra \mathcal{G} on M can be lifted to the action of a connected Lie group G (such that $Lie(G) = \mathcal{G}$) we speak about global Poisson-Lie symmetry. G will be then referred to as the symmetry group of (M, μ) and B as the cosymmetry group. Note that G acts on M and

B underlies the way how this action is expressed via the Poisson brackets. If there is distinguished (evolution) vector field $v \in Vect(M)$ leaving invariant $Im(\mu^*)$, we say that the dynamical system $(M, \{.,.\}, v)$ is (G, B) -Poisson-Lie symmetric (cf. [11]). We also note that $y \in Fun(B)$ can be interpreted as a non-Abelian (or Poisson-Lie) Hamiltonian of the vector field $w_\mu(y)$. The fact that w_μ is anti-homomorphism just implies a nice formula $[w_\mu(x), w_\mu(y)] = -w_\mu(\{x, y\}_B)$. If the group B is Abelian then $\Delta(x) = 1 \otimes x + x \otimes 1$ and (3) is nothing but the standard Hamiltonian formula $w_\mu(y)f = \{f, \mu^*(y)\}$. Thus the Poisson-Lie symmetry becomes the standard Hamiltonian symmetry if the cosymmetry group B is Abelian.

Let us note also that the Definition 1 can be reformulated by using the Maurer-Cartan form ρ_B and thus avoiding to refer to the coproduct on $Fun(B)$ (this essentially amounts to replace $dy'S(y'')$ by $\langle \rho_B, V \rangle$). There are two reasons that we choose the formulation that uses the coproduct and the antipode. First one is not directly related to this paper, but is important in general in perspective of quantization. Indeed, for the definition of the Hopf symmetry the notions of coproduct and antipode are indispensable already at the level of basic definition and the close relationship between the Poisson-Lie and Hopf symmetry thus becomes more transparent. The second reason is more practical. In fact, the notation using the coproduct and the antipode is technically more convenient in elaborating and formulating proofs of the theorems presented in the paper.

Remark: Our definition of the Poisson-Lie symmetry and the traditional one are close cousins but they are not quite identical. For example, a traditional symmetry must admit a moment map in order to be the symmetry in the new sense and the newly defined symmetry must be global in order to be traditional. The main reason why we shall use the new definition is its usefulness for treatment of anomalies which cause obstructions for gauging the Poisson-Lie symmetries. The traditional definition does not see the difference between anomalous and non-anomalous cases while the new definition gives the very simply criterion to distinguish them. In what follows, we shall work exclusively with the new definition and we hope to convince the reader about its naturalness and usefulness.

Lemma 2: Every Poisson map $\mu : M \rightarrow B$ realizes the Poisson-Lie symmetry of M .

Proof: First remind that the map $\mu : M \rightarrow B$ is a Poisson morphism iff the dual map $\mu^* : Fun(B) \rightarrow Fun(M)$ satisfies

$$\{\mu^*(x), \mu^*(y)\} = \mu^*({x, y}_B), \quad x, y \in Fun(B). \quad (4)$$

Now we take $x, y \in Fun(B)$ and calculate

$$\begin{aligned} [w_\mu(y), w_\mu(x)]f &= \\ \{\{f, \mu^*(x')\}\mu^*(S(x''))\}, \mu^*(y')\}\mu^*(S(y'')) - \{\{f, \mu^*(y')\}\mu^*(S(y''))\}, \mu^*(x')\}\mu^*(S(x'')) &= \\ = \{f, \{\mu^*(x'), \mu^*(y')\}\}S(\mu^*(x''))S(\mu^*(y'')) - \{f, \mu^*(x'y')\}\{\mu^*(S(x'')), \mu^*(S(y''))\} &= \\ = \{f, \mu^*({x, y}'_B)\}\mu^*(S({x, y}''_B)) = w_\mu({x, y}_B)f \end{aligned}$$

Going from the second to the third line we have used the Jacobi identity and the fact that $x'S(x'')$ is a number (the count of x). We have passed from the third to the fourth line by using (1),(2ab) and (4).

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2.3 Anomalous realizations

The Poisson-Lie symmetry can be realized also by a map $\mu : M \rightarrow B$ which is not the Poisson morphism. If this happens we speak about the anomalous Poisson-Lie symmetry and we call μ the anomalous moment map. Anomalous moment maps naturally arise by twisting the Heisenberg doubles. The detailed exposition of this fact will be our following subject.

Definition 2: Let D be an even-dimensional Lie group equipped with a maximally Lorentzian bi-invariant metric. If $Lie(D) = Lie(G) \dot{+} Lie(B)$, where G and B are maximally isotropic subgroups, D is called the Drinfeld double of G or the Drinfeld double of B . Let κ be a metric preserving automorphism of D and suppose that there are respective basis T^i and t_i ($i = 1, \dots, n$) of $\mathcal{G} = Lie(G)$ and $\mathcal{B} = Lie(B)$ such that

$$(T^i, t_j)_D = \delta_j^i. \quad (5)$$

Then the (basis independent) expression

$$\{f_1, f_2\}_D \equiv \nabla_{T^i}^R f_1 \nabla_{t_i}^R f_2 - \nabla_{\kappa(t_i)}^L f_1 \nabla_{\kappa(T^i)}^L f_2, \quad f_1, f_2 \in Fun(D) \quad (6)$$

is a Poisson bracket and the Poisson manifold $(D, \{.,.\}_D)$ is called the twisted Heisenberg double.

Theorem 1: Let D be a twisted Heisenberg double which is also decomposable, i.e. such that two global unambiguous decompositions hold: $D = \kappa(B)G$ and $D = \kappa(G)B$. Consider (smooth) maps $\Lambda_L, \Lambda_R : D \rightarrow B$, $\Xi_R, \Xi_L : D \rightarrow G$ respectively induced by these two decompositions. Then it holds:

- a) The Poisson manifold $(D, \{.,.\}_D)$ is symplectic.
- b) Both maps Λ_L and Λ_R realize the global (anomalous) Poisson-Lie symmetries of the symplectic manifold $(D, \{.,.\}_D)$. The corresponding symmetry group is G acting as

$$h \triangleright K = \kappa(h)K, \quad h \in G, \quad K \in D, \quad (7a)$$

or, respectively, as

$$h \triangleright K = Kh^{-1}, \quad h \in G, \quad K \in D. \quad (7b)$$

Explanations: The symbol $+$ stands for the direct sum of vector spaces only and not of Lie algebras. Bi-invariant means both left- and right-invariant. The non-degenerated bi-invariant metric on D obviously induces an Ad -invariant non-degenerated bilinear form $(.,.)_{\mathcal{D}}$ on $\mathcal{D} = Lie(D)$. An isotropic submanifold of D is such that the induced metric on it vanishes. Maximally isotropic means that it is not contained in any bigger isotropic submanifold. The vector fields $\nabla_T^{L,R}$ are defined as

$$\nabla_T^L f(K) \equiv \delta_T(f')f''(K) = \left(\frac{d}{ds} \right)_{s=0} f(e^{sT}K),$$

$$\nabla_T^R f(K) \equiv \delta_T(f'')f'(K) = \left(\frac{d}{ds} \right)_{s=0} f(Ke^{sT}),$$

where $f \in Fun(D)$, $K \in D$, $T \in Lie(D)$. Global unambiguous decomposition $D = \kappa(B)G$ means that for every element $K \in D$ it exists a unique $g = \Xi_R(K) \in G$ and a unique $b = \Lambda_L(K) \in B$ such that $K = \kappa(b)g^{-1}$. Similarly for $D = \kappa(G)B$: it exists a unique $\tilde{g} = \Xi_L(K) \in G$ and a unique $\tilde{b} = \Lambda_R(K) \in B$ such that $K = \kappa(\tilde{g})\tilde{b}^{-1}$. The fact that the formula (6) defines

the Poisson bracket was proved by Semenov-Tian-Shansky in [16] and, for completeness, we shall outline here his argument:

Consider a (basis independent) element $c \in \mathcal{D} \otimes \mathcal{D}$ given by

$$c = T^i \otimes t_i + t_i \otimes T^i.$$

It is easy to see that the Ad -invariance and κ -invariance of the bilinear form $(\cdot, \cdot)_{\mathcal{D}}$ implies the Ad -invariance and κ -invariance of c . Thus the bracket (6) can be rewritten as

$$\{f_1, f_2\}_D = \frac{1}{2} \nabla_{T^i}^R f_1 \nabla_{t_i}^R f_2 - \frac{1}{2} \nabla_{t_i}^R f_1 \nabla_{T^i}^R f_2 + \frac{1}{2} \nabla_{\kappa(T^i)}^L f_1 \nabla_{\kappa(t_i)}^L f_2 - \frac{1}{2} \nabla_{\kappa(t_i)}^L f_1 \nabla_{\kappa(T^i)}^L f_2.$$

Note that in this bracket appear two elements of $\mathcal{D} \wedge \mathcal{D}$ given by

$$r_{\mathcal{D}} = \frac{1}{2} T^i \otimes t_i - \frac{1}{2} t_i \otimes T^i, \quad r_{\mathcal{D}}^{\kappa} = \frac{1}{2} \kappa(T^i) \otimes \kappa(t_i) - \frac{1}{2} \kappa(t_i) \otimes \kappa(T^i).$$

It can be shown by direct calculation that the algebraic Schouten brackets $[r_{\mathcal{D}}, r_{\mathcal{D}}]_S$ (cf. [9], Eqs. (4.36-39)) gives an invariant element of $\wedge^3 \mathcal{D}$ and, moreover, $[r_{\mathcal{D}}^{\kappa}, r_{\mathcal{D}}^{\kappa}]_S = [r_{\mathcal{D}}, r_{\mathcal{D}}]_S$. Those facts imply that the Semenov-Tian-Shansky bracket (6) satisfies the Jacobi identity.

Let us finish the Explanations by saying that the list of decomposable doubles is not very long. The typical examples are the cotangent bundle T^*G of any Lie group G , the complexification $G^{\mathbb{C}}$ of a compact (loop) group G and certain Drinfeld twists of two first items. Nevertheless, the independent theorem dealing with decomposable doubles is useful for two reasons. First of them is the range of applicability: many resolvable quantum theories have compact (quantum) group symmetry and in this or other way are based on the short list of decomposable doubles. The other reason is that the notion of the Poisson-Lie symmetry is traditionally globally defined and the decomposable doubles lead to global Poisson-Lie symmetry. Let us stress, however, that the local Poisson-Lie symmetries must be considered equally seriously (for instance the conformal symmetry in field theory is only local but physically relevant). This is the reason that we devote the section 3.3 to non-decomposable doubles where the number of examples is very big.

Proof of Theorem 1:

a): Consider a point $K \in D$ and four linear subspaces of the tangent space $T_K D$ defined as $S_L = L_{K*}\mathcal{G}$, $S_R = R_{K*}\kappa(\mathcal{G})$, $\tilde{S}_L = L_{K*}\mathcal{B}$ and $\tilde{S}_R = R_{K*}\kappa(\mathcal{B})$. (The symbols L_{K*} and R_{K*} stand for left and right transport on the group D , respectively). The existence of the global decompositions $D = \kappa(B)G$ and $D = \kappa(G)B$ means that at every $K \in D$ the tangent space $T_K D$ can be decomposed as $T_K D = S_L + \tilde{S}_R$ and $T_K D = \tilde{S}_L + S_R$, respectively. This fact makes possible to introduce a projector $\Pi_{L\tilde{R}}$ on \tilde{S}_R with a kernel S_L and a projector $\Pi_{\tilde{L}R}$ on S_R with a kernel \tilde{S}_L . At every point $K \in D$ we can therefore define a following 2-form ω

$$\omega(t, u) = (t, (\Pi_{\tilde{L}R} - \Pi_{L\tilde{R}})u)_{\mathcal{D}}, \quad (8)$$

where t, u are arbitrary vectors in $T_K D$ and $(\cdot, \cdot)_{\mathcal{D}}$ is the bi-invariant metric at the point K (it is related by the left or right transport of the Ad-invariant bilinear form $(\cdot, \cdot)_{\mathcal{D}}$ defined at the unit element $E \in D$). Let us show that ω is the symplectic form corresponding to the Poisson structure $\{.,.\}_{\mathcal{D}}$. First of all we remark that the Poisson bivector (=contravariant antisymmetric tensor) corresponding to the Poisson bracket $\{.,.\}_{\mathcal{D}}$ reads

$$\alpha = L_{K*}(T^i \otimes t_i) - R_{K*}(\kappa(t_i) \otimes \kappa(T^i)). \quad (9)$$

Introduce two more projectors $\Pi_{R\tilde{R}}, \Pi_{\tilde{L}L}$, where the first subscript stands for the kernel and the second for the image. Then we conclude

$$\begin{aligned} \alpha(., \omega(., u)) &= \\ &= L_{K*}T^i(L_{K*}t_i, (\Pi_{\tilde{L}R} - \Pi_{L\tilde{R}})u)_{\mathcal{D}} - R_{K*}\kappa(t_i)(R_{K*}\kappa(T^i), (\Pi_{\tilde{L}R} - \Pi_{L\tilde{R}})u)_{\mathcal{D}} = \\ &= (\Pi_{\tilde{L}L} - \Pi_{R\tilde{R}})(\Pi_{\tilde{L}R} - \Pi_{L\tilde{R}})u = u. \end{aligned} \quad (10)$$

Proof of b) and c) Consider a bracket $\{.,.\}_B$ on the cosymmetry group B given by

$$\{x, y\}_B(b) = -(T^i, Ad_b T^k)_{\mathcal{D}}(\nabla_{t_i}^L x)(b)(\nabla_{t_k}^R y)(b), \quad b \in B, \quad x, y \in Fun(B). \quad (11)$$

It was shown in Proposition 4.5. of [9] that $\{.,.\}_B$ is the Poisson-Lie bracket on B . We shall prove that

$$\{\Lambda_L^*(x), \Lambda_L^*(y)\}_D = \Lambda_L^*\left(\{x, y\}_B - M_{\kappa}^{ij} \nabla_{t_i}^R x \nabla_{t_j}^R y\right), \quad x, y \in Fun(B), \quad (12a)$$

$$\{\Lambda_R^*(x), \Lambda_R^*(y)\}_D = \Lambda_R^*\left(\{x, y\}_B - M_{\kappa^{-1}}^{ij} \nabla_{t_i}^R x \nabla_{t_j}^R y\right), \quad x, y \in Fun(B), \quad (12b)$$

where the constant antisymmetric matrix M_{κ}^{ij} is given by

$$M_{\kappa} = Q_{\kappa} P_{\kappa}^{-1}, \quad (P_{\kappa})_i{}^j = (\kappa(t_i), T^j)_{\mathcal{D}}, \quad Q_{\kappa}^{ij} = (\kappa(T^i), T^j)_{\mathcal{D}}. \quad (13)$$

We note that the non-degeneracy of $(\cdot, \cdot)_{\mathcal{D}}$ and also the global decomposabilities $D = \kappa(B)G = \kappa(G)B$ guarantee that both matrices P_{κ} and $P_{\kappa^{-1}}$ are invertible.

In order to calculate the bracket $\{\Lambda_L^*(x), \Lambda_L^*(y)\}_D$, we use the defining formula (6). We first realize that

$$\nabla_{T^i}^R \Lambda_L^*(x) = \left(\frac{d}{ds}\right)_{s=0} x(\Lambda_L(K e^{sT^i})) = 0 \quad (14)$$

and then we write

$$\begin{aligned} \{\Lambda_L^*(x), \Lambda_L^*(y)\}_D &= -\nabla_{\kappa(t_i)}^L \Lambda_L^*(x) \nabla_{\kappa(T^i)}^L \Lambda_L^*(y) = \\ &= -\left(\frac{d}{ds_1}\right)_{s_1=0} x(\Lambda_L(e^{s_1 \kappa(t_i)} K)) \left(\frac{d}{ds_2}\right)_{s_2=0} y(\Lambda_L(e^{s_2 \kappa(T^i)} K)) = \\ &= -\Lambda_L^*(\nabla_{t_i}^L x) \left(\frac{d}{ds_2}\right)_{s_2=0} y(\Lambda_L(e^{s_2 \kappa(T^i)} \kappa(\Lambda_L(K)))) = \\ &= -\Lambda_L^*(\nabla_{t_i}^L x) \left(\frac{d}{ds}\right)_{s=0} y(\Lambda_L(\kappa[\Lambda_L(K) \exp(s \Lambda_L^{-1}(K) T^i \Lambda_L(K))])). \end{aligned} \quad (15)$$

We note that

$$\Lambda_L^{-1}(K) T^i \Lambda_L(K) = (\Lambda_L^{-1}(K) T^i \Lambda_L(K), t_k)_{\mathcal{D}} T^k + (\Lambda_L^{-1}(K) T^i \Lambda_L(K), T^k)_{\mathcal{D}} t_k.$$

This identity permits to rewrite the r.h.s. of (15) as the sum of two terms

$$\{\Lambda_L^*(x), \Lambda_L^*(y)\}_D = V_1 + V_2,$$

where

$$V_1 = -(\Lambda_L^{-1}(K) T^i \Lambda_L(K), T^k)_{\mathcal{D}} \Lambda_L^*(\nabla_{t_i}^L x) \Lambda_L^*(\nabla_{t_k}^R y) = \Lambda_L^*(\{x, y\}_B)$$

and

$$V_2 = -(\Lambda_L^{-1}(K)T^i\Lambda_L(K), t_k)_{\mathcal{D}}\Lambda_L^*({}^B\nabla_{t_i}^L x)\left(\frac{d}{ds}\right)_{s=0}y(\Lambda_L(\kappa[\Lambda_L(K)\exp(sT^k)])) =$$

$$-\Lambda_L^*({}^B\nabla_{t_k}^R x)\left(\frac{d}{ds}\right)_{s=0}y(\Lambda_L(\kappa[\Lambda_L(K)\exp(sT^k)])) = -\Lambda_L^*({}^B\nabla_{t_k}^R x)\Lambda_L^*({}^B\nabla_{\tau^k}^R y).$$

The element $\tau^k \in \mathcal{B}$ is defined by the $D = \kappa(B)G$ decomposition

$$\kappa(T^k) = \kappa(\tau^k) + c^k, \quad c^k \in \mathcal{G}.$$

From this it is easy to find that

$$\tau^k = M_{\kappa}^{kl}t_l,$$

where the matrix M_{κ} was introduced in (13). Putting all together, we arrive at

$$\{\Lambda_L^*(x), \Lambda_L^*(y)\}_D = \Lambda_L^*\left(\{x, y\}_B - M_{\kappa}^{ij}{}^B\nabla_{t_i}^R x {}^B\nabla_{t_j}^R y\right),$$

which is nothing but (12a). The identity (12b) can be proved in a similar way. We note also that our notation has distinguished the invariant derivatives on $Fun(D)$ and on $Fun(B)$ (the derivatives on $Fun(B)$ were denoted as ${}^B\nabla^{R,L}$). We shall not make this distinction in what follows and we let the reader to understand from the context on which space $\nabla^{R,L}$ act.

In case where the twisting automorphism is trivial (i.e. κ is identity), the anomaly matrices M_{κ} , $M_{\kappa^{-1}}$ vanish and $\Lambda_{L,R} : D \rightarrow B$ are the Poisson maps. From Lemma 2 it then follows that $\Lambda_{L,R} : D \rightarrow B$ realize the Poisson-Lie symmetries of D . Let us show now that in the case of non-trivial twisting the maps $\Lambda_{L,R} : D \rightarrow B$ also realize the Poisson-Lie symmetries although they are not Poisson morphisms. For this, we first remind the definition (3) of the map $w_{\Lambda_L} : Fun(B) \rightarrow Vect(D)$:

$$w_{\Lambda_L}(x)f = \{f, \Lambda_L^*(x')\}_D \Lambda_L^*(S(x'')), \quad x \in Fun(B), \quad f \in Fun(D).$$

We calculate

$$(w_{\Lambda_L}(y)w_{\Lambda_L}(x) - w_{\Lambda_L}(x)w_{\Lambda_L}(y))f =$$

$$= \{\{f, \Lambda_L^*(x')\}_D \Lambda_L^*(S(x'')), \Lambda_L^*(y')\}_D \Lambda_L^*(S(y'')) - (x \leftrightarrow y) =$$

$$\begin{aligned}
&= \{\{f, \Lambda_L^*(x')\}_D \Lambda_L^*(y')\}_D \Lambda_L^*(S(x''y'')) + \{f, \Lambda_L^*(x')\}_D \{\Lambda_L^*(S(x'')), \Lambda_L^*(y')\}_D \Lambda_L^*(S(y'')) \\
&\quad - (x \leftrightarrow y) = \\
&= \{f, \{\Lambda_L^*(x'), \Lambda_L^*(y')\}_D\}_D \Lambda_L^*(S(x''y'')) - \{f, \Lambda_L^*(x'y')\}_D \{\Lambda_L^*(S(x'')), \Lambda_L^*(S(y''))\}_D.
\end{aligned}$$

Now we use the formula (12a) and the Poisson-Lie property (1) of the bracket $\{.,.\}_B$ to obtain

$$\begin{aligned}
&[w_{\Lambda_L}(y), w_{\Lambda_L}(x)]f = \\
&= \{f, \Lambda_L^*(\{x', y'\}_B)\}_D \Lambda_L^*(S(x''y'')) - \{f, \Lambda_L^*(x'y')\}_D \Lambda_L^*(\{S(x''), S(y'')\}_B) + \\
&- M_{\kappa}^{ij} \left(\{f, \Lambda_L^*(\nabla_{t_i}^R x' \nabla_{t_j}^R y')\}_D \Lambda_L^*(S(x''y'')) - \{f, \Lambda_L^*(x'y')\}_D \Lambda_L^*(\nabla_{t_i}^R S(x'') \nabla_{t_j}^R S(y'')) \right)
\end{aligned}$$

The last line of this expression vanishes due to following identities

$$(\nabla_{t_i}^R y') S(y'') + y' \nabla_{t_i}^R S(y'') = \nabla_{t_i}^R (y' S(y'')) = 0,$$

$$(\nabla_{t_i}^R \nabla_{t_i}^L x') S(x'') + \nabla_{t_i}^L x' \nabla_{t_i}^R S(x'') = \nabla_{t_i}^R (\nabla_{t_i}^L x' S(x'')) = 0$$

and (using (6))

$$\begin{aligned}
&\{f, \Lambda_L^*(\nabla_{t_i}^R x')\}_D \Lambda_L^*(S(x'')) + \{f, \Lambda_L^*(x')\}_D \Lambda_L^*(\nabla_{t_i}^R S(x'')) = \\
&= \nabla_{\kappa(T^i)}^L f \Lambda_L^*((\nabla_{t_i}^R \nabla_{t_i}^L x') S(x'') + \nabla_{t_i}^L x' \nabla_{t_i}^R S(x'')) = 0.
\end{aligned}$$

Now we use the Poisson-Lie properties (1),(2) to arrive at

$$\begin{aligned}
&[w_{\Lambda_L}(y), w_{\Lambda_L}(x)]f = \\
&= \{f, \Lambda_L^*(\{x', y'\}_B)\}_D \Lambda_L^*(S(x''y'')) + \{f, \Lambda_L^*(x'y')\}_D \Lambda_L^*(S(\{x'', y''\}_B)) = \\
&= w_{\Lambda_L}(\{x, y\}_B) f.
\end{aligned}$$

According to the Definition 1, the map Λ_L thus realizes the Poisson-Lie symmetry of D .

Much in the same way, we obtain also

$$[w_{\Lambda_R}(y), w_{\Lambda_R}(x)]f = w_{\Lambda_R}(\{x, y\}_B) f,$$

where

$$w_{\Lambda_R}(x) f = \{f, \Lambda_R^*(x')\}_D \Lambda_R^*(S(x'')), \quad x \in Fun(B), f \in Fun(D).$$

Having established that both maps $w_{\Lambda_L}, w_{\Lambda_R} : Fun(B) \rightarrow Vect(D)$ are Lie algebra homomorphisms (i.e. that both $\Lambda_L, \Lambda_R : D \rightarrow B$ realize Poisson-Lie symmetries), it remains to find what are the corresponding symmetry groups. We use (6) and (0) to obtain

$$w_{\Lambda_L}(y)f = \{f, \Lambda_L^*(y')\} \Lambda_L^*(S(y'')) = \nabla_{\kappa(T^i)}^L f \Lambda_L^*((\nabla_{t_i}^L y')S(y'')) = \delta_{t_i}(y) \nabla_{\kappa(T^i)}^L f. \quad (16a)$$

We remind that δ_{t_i} is the ε -derivative (cf. Sec. 2.1) hence $\delta_{t_i}(y)$ is a real number for every i . It therefore follows that $Im(w_{\Lambda_L}) = \kappa(\mathcal{G})$ and we have proved (7a). Similarly, we obtain

$$w_{\Lambda_R}(y)f = -\delta_{t_i}(y) \nabla_{T^i}^R f, \quad (16b)$$

which proves (7b).

#

3 Non-anomalous moment maps

Non-anomalous Poisson-Lie symmetries play very important role in the symplectic geometry since they permit to perform the so called symplectic reduction (or "gauging" in the terminology of physicists). However, given a decomposable twisted Heisenberg double (D, κ) , the basic moment maps Λ_L, Λ_R are generically anomalous and cannot be gauged. Indeed, the anomaly matrices $M_{\kappa}^{ij}, M_{\kappa^{-1}}^{ij}$ vanish only in the case where the twisting automorphism κ preserves the symmetry group G (cf. (13)). In this section, we shall look for other moment maps (distinct from Λ_L, Λ_R) which would allow us to gauge (D, κ) . It turns out, that the existence of the non-anomalous Poisson-Lie moment maps associated to the twisted Heisenberg double heavily depend on the details of the structure of (D, κ) . In the three following subsections, we shall discuss three interesting cases, where the non-anomalous moment maps can be constructed. We shall keep the exposition of the two first cases (a quasi-adjoint action and a proper subsymmetry) in an abstract level since the concrete examples will be discussed in the subsequent Section 4. However, we shall illustrate the third case (an improper subsymmetry) already in this Section 3, since later we shall not consider it anymore.

3.1 Quasi-adjoint action

In this subsection, we shall consider the decomposable twisted Heisenberg doubles for which the twisting automorphism κ preserves the cosymmetry group B . We have the following theorem:

Theorem 2: Let D be a decomposable twisted Heisenberg double such that the twisting automorphism κ preserves the subgroup B . Consider the anomalous moment maps Λ_L, Λ_R and construct two new maps $B_L : D \rightarrow B$ and $B_R : D \rightarrow B$ as follows

$$B_L(K) = \kappa(\Lambda_L(K))\Lambda_R(K), \quad B_R(K) = \kappa^{-1}(\Lambda_R(K))\Lambda_L(K), \quad K \in D.$$

Then it holds: Both maps B_L and B_R are Poisson and they realize global non-anomalous Poisson-Lie symmetries of $(D, \{.,.\}_D)$. The corresponding symmetry group is G acting as

$$h \triangleright K = \kappa(h)K\Xi_R(\kappa[h\Lambda_L(K)]), \quad h \in G, \quad K \in D,$$

or, respectively, as

$$h \triangleright K = \kappa[\Xi_L^{-1}(\Lambda_R^{-1}(K)h^{-1})]Kh^{-1}. \quad h \in G, \quad K \in D.$$

Proof: Consider two functions $x, y \in Fun(B)$. We know already that it holds

$$\{\Lambda_L^*(x), \Lambda_L^*(y)\}_D = \Lambda_L^*\left(\{x, y\}_B - M_\kappa^{ij} \nabla_{t_i}^R x \nabla_{t_j}^R y\right), \quad x, y \in Fun(B), \quad (12a)$$

$$\{\Lambda_R^*(x), \Lambda_R^*(y)\}_D = \Lambda_R^*\left(\{x, y\}_B - M_{\kappa^{-1}}^{ij} \nabla_{t_i}^R x \nabla_{t_j}^R y\right), \quad x, y \in Fun(B), \quad (12b)$$

where the Poisson-Lie bracket $\{.,.\}_B$ and matrices $M_\kappa, M_{\kappa^{-1}}$ were defined in (11) and in (13), respectively. Introduce maps $\Gamma_L : D \rightarrow B, \Gamma_R : D \rightarrow B$ by

$$\Gamma_L(K) = \kappa(\Lambda_L(K)), \quad \Gamma_R(K) = \kappa^{-1}(\Lambda_R(K)), \quad K \in D$$

hence $B_L = \Gamma_L \Lambda_R$ and $B_R = \Gamma_R \Lambda_L$. We shall now prove that

$$\{\Gamma_L^*(x), \Gamma_L^*(y)\}_D = \Gamma_L^*\left(\{x, y\}_B + M_{\kappa^{-1}}^{ij} \nabla_{t_i}^L x \nabla_{t_j}^L y\right), \quad x, y \in Fun(B), \quad (17a)$$

$$\{\Gamma_R^*(x), \Gamma_R^*(y)\}_D = \Gamma_R^*\left(\{x, y\}_B + M_\kappa^{ij} \nabla_{t_i}^L x \nabla_{t_j}^L y\right), \quad x, y \in Fun(B). \quad (17b)$$

First we remark that

$$\begin{aligned} (\nabla_{T^i}^R \Lambda_L^*(x))(K) &= \left(\frac{d}{ds}\right)_{s=0} x\left((\Lambda_L(K e^{sT^i}))\right) = 0, \quad K \in D, \\ (\nabla_{\kappa(T^i)}^L \Gamma_R^*(y))(K) &= \left(\frac{d}{ds}\right)_{s=0} y\left(\kappa^{-1}(\Lambda_R(e^{s\kappa(T^i)} K))\right) = 0, \quad K \in D. \end{aligned}$$

Thus, using the fundamental definition (6), we obtain

$$\{\Lambda_L^*(x), \Gamma_R^*(y)\}_D = 0$$

and

$$\begin{aligned} &\{\Gamma_R^*(x), \Gamma_R^*(y)\}_D = \\ &= \left(\frac{d}{ds_1}\right)_{s_1=0} x\left(\kappa^{-1}(\Lambda_R(K e^{s_1 T^i}))\right) \left(\frac{d}{ds_2}\right)_{s_2=0} y\left(\kappa^{-1}(\Lambda_R(K e^{s_2 T^i}))\right) = \\ &= -\left(\frac{d}{ds}\right)_{s=0} x\left(\kappa^{-1}(\Lambda_R(K e^{sT^i}))\right) \Gamma_R^*\left(\nabla_{\kappa^{-1}(t_i)}^L y\right) = \\ &= \Gamma_R^*\left((b^{-1} \kappa^{-1}(T^i) b, T^j)_{\mathcal{D}} \nabla_{t_j}^R x \nabla_{\kappa^{-1}(t_i)}^L y\right) = \\ &= \Gamma_R^*\left(\left[(b^{-1} T^i b, T^j)_{\mathcal{D}} - (T^i, \kappa^{-1}(T^m))_{\mathcal{D}} (b T^j b^{-1}, t_l)_{\mathcal{D}} (T^l, \kappa^{-1}(t_m))_{\mathcal{D}}\right] \nabla_{t_j}^R x \nabla_{t_i}^L y\right) = \\ &= \Gamma_R^*\left(\{x, y\}_B + (T^i, \kappa^{-1}(T^m))_{\mathcal{D}} (\kappa^{-1}(t_m), T^j)_{\mathcal{D}} \nabla_{t_i}^L x \nabla_{t_j}^L y\right) = \\ &= \Gamma_R^*\left(\{x, y\}_B + M_\kappa^{ij} \nabla_{t_i}^L x \nabla_{t_j}^L y\right). \end{aligned} \quad (17b)$$

We note that $b \in B$ in this formula denotes the argument of functions in $Fun(B)$. Similarly, we can prove that

$$\{\Lambda_R^*(x), \Gamma_L^*(y)\}_D = 0$$

and

$$\{\Gamma_L^*(x), \Gamma_L^*(y)\}_D = \Gamma_L^*\left(\{x, y\}_B + M_{\kappa^{-1}}^{ij} \nabla_{t_i}^L x \nabla_{t_j}^L y\right), \quad x, y \in Fun(B), \quad (17a)$$

Now we calculate

$$\begin{aligned}
\{B_L^*(x), B_L^*(y)\}_D &= \{\Gamma_L^*(x')\Lambda_R(x''), \Gamma_L^*(y')\Lambda_R(y'')\}_D = \\
&= \{\Gamma_L^*(x'), \Gamma_L^*(y')\}\Lambda_R(x'')\Lambda_R(y'') + \Gamma_L^*(x')\Gamma_L^*(y')\{\Lambda_R(x''), \Lambda_R(y'')\} = \\
&\quad \Gamma_L^*\left(\{x', y'\}_B + M_{\kappa^{-1}}^{ij}\nabla_{t_i}^L x' \nabla_{t_j}^L y'\right)\Lambda_R(x'')\Lambda_R(y'') + \\
&\quad + \Gamma_L^*(x')\Gamma_L^*(y')\Lambda_R^*\left(\{x'', y''\}_B - M_{\kappa^{-1}}^{ij}\nabla_{t_i}^R x'' \nabla_{t_j}^R y''\right) = \\
&= B_L^*\left(\{x, y\}_B + M_{\kappa^{-1}}^{ij}\nabla_{t_i}^L x \nabla_{t_j}^L y - M_{\kappa^{-1}}^{ij}\nabla_{t_i}^R x \nabla_{t_j}^R y\right), \tag{18a}
\end{aligned}$$

Similarly, we obtain

$$\{B_R^*(x), B_R^*(y)\}_D = B_R^*\left(\{x, y\}_B + M_{\kappa}^{ij}\nabla_{t_i}^L x \nabla_{t_j}^L y - M_{\kappa}^{ij}\nabla_{t_i}^R x \nabla_{t_j}^R y\right). \tag{18b}$$

The reader may be surprised by the presence of the anomaly matrices M_{κ} , $M_{\kappa^{-1}}$ in the resulting formulas (18a) and (18b). Didn't we promise that the moment maps B_L, B_R realize non-anomalous Poisson-Lie symmetries? Well, the point is the following: If the twisting automorphism κ preserves the cosymmetry group B then there are three natural Poisson-Lie brackets on $Fun(B)$. The first one is evident; it is given by the formula (11) of Section 2.3:

$$\{x, y\}_B(b) = -(T^i, Ad_b T^k)_{\mathcal{D}}(\nabla_{t_i}^L x)(b)(\nabla_{t_k}^R y)(b), \quad b \in B, \quad x, y \in Fun(B).$$

The second and the third bracket are defined by

$$\{x, y\}_B^{\kappa}(b) = -(\kappa(T^i), Ad_b \kappa(T^k))_{\mathcal{D}}(\nabla_{\kappa(t_i)}^L x)(b)(\nabla_{\kappa(t_k)}^R y)(b), \tag{19a}$$

$$\{x, y\}_B^{\kappa^{-1}}(b) = -(\kappa^{-1}(T^i), Ad_b \kappa^{-1}(T^k))_{\mathcal{D}}(\nabla_{\kappa^{-1}(t_i)}^L x)(b)(\nabla_{\kappa^{-1}(t_k)}^R y)(b). \tag{19b}$$

It is easy to understand why the brackets (19a) and (19b) verify the Jacobi identity and the Poisson-Lie property (1). It is because they appear on the same footing as the original bracket (11). Indeed, the double D is not only the double of the pair of groups G and B , but it is also the double of the pair $\kappa(G)$ and $\kappa(B) = B$ and of the pair $\kappa^{-1}(G)$ and $\kappa^{-1}(B) = B$. Each of the three pairs generate the respective basis T^i, t_i ; $\kappa(T^i), \kappa(t_i)$ and $\kappa^{-1}(T^i), \kappa^{-1}(t_i)$, all three basis sharing the crucial duality property (5).

The brackets (19a) and (19b) can be worked out in the basis t_i instead of $\kappa(t_i)$ or $\kappa^{-1}(t_i)$. We use obvious identities

$$\kappa(t^i) = (\kappa(t^i), T^m)_{\mathcal{D}} t_m, \quad \kappa^{-1}(t^i) = (\kappa^{-1}(t^i), T^m)_{\mathcal{D}} t_m$$

and we find

$$\begin{aligned} \{x, y\}_B^\kappa &= \{x, y\}_B + M_{\kappa^{-1}}^{ij} \nabla_{t_i}^L x \nabla_{t_j}^L y - M_{\kappa^{-1}}^{ij} \nabla_{t_i}^R x \nabla_{t_j}^R y, \\ \{x, y\}_B^{\kappa^{-1}} &= \{x, y\}_B + M_\kappa^{ij} \nabla_{t_i}^L x \nabla_{t_j}^L y - M_\kappa^{ij} \nabla_{t_i}^R x \nabla_{t_j}^R y. \end{aligned}$$

This permits us to rewrite (18a) and (18b) as

$$\begin{aligned} \{B_L^*(x), B_L^*(y)\}_D &= B_L^* \left(\{x, y\}_B^\kappa \right), \\ \{B_R^*(x), B_R^*(y)\}_D &= B_R^* \left(\{x, y\}_B^{\kappa^{-1}} \right). \end{aligned}$$

We thus conclude that the moment maps B_L and B_R are indeed non-anomalous with respect to the Poisson-Lie brackets (19a) and (19b).

Every Poisson-Lie moment map μ generates the action of the Lie algebra \mathcal{G} and, in good cases, this \mathcal{G} -action can be lifted to the action of the symmetry group G . Let us now show that the moment maps B_L, B_R are those "good" cases yielding the global non-anomalous Poisson-Lie symmetries. The following exposition uses some standard conventions concerning the Hopf algebra calculations (see [8]), namely, the repeated application of the coproduct is written as

$$(\Delta \otimes Id \otimes Id)(\Delta \otimes Id)\Delta(x) \equiv x' \otimes x'' \otimes x''' \otimes x''', \quad x \in Fun(B).$$

The reader has certainly noticed that this is the generalization of the Sweedler notation introduced in Section 2.1.

Consider first a set of functions $x^i \in Fun(B)$ which is dual to the basis t_i of $\mathcal{B} = Lie(B)$, i.e. it holds

$$\delta_{t_j}(x^i) = \delta_j^i,$$

where δ_{t_j} are the ε -derivatives. We denote by $\kappa(x^i)$ the functions on B of the form

$$\kappa(x^i)(b) = x^i(\kappa(b)), \quad b \in B.$$

We are going to make explicit the basic map $w_{B_L} : Fun(B) \rightarrow Vect(D)$ expressing the action of $Lie(G)$ on $f \in Fun(D)$ (cf. (3)).

$$\begin{aligned}
w_{B_L}(\kappa^{-1}(x^i))f &= \{f, B_L^*(\kappa^{-1}((x^i)'))\}B_L^*(S(\kappa^{-1}((x^i)''))) = \\
&= \{f, \Gamma_L^*(\kappa^{-1}((x^i)'))\Lambda_R^*(\kappa^{-1}((x^i)''))\}\Gamma_L^*(S(\kappa^{-1}((x^i)'''')))\Lambda_R^*(S(\kappa^{-1}((x^i)''''))) = \\
&= \nabla_{\kappa(T^i)}^L f - \delta_{t_k}(\kappa^{-1}((x^i)''))\Gamma_L^*(\kappa^{-1}((x^i)'))\Gamma_L^*(S(\kappa^{-1}((x^i)'''')))\nabla_{T^k}^R f = \\
&= \nabla_{\kappa(T^i)}^L f - (\Gamma_L(K)t_k\Gamma_L^{-1}(K), \kappa(T^i))_{\mathcal{D}}\nabla_{T^k}^R f = \\
&= \nabla_{\kappa(T^i)}^L f - (\Lambda_L(K)\kappa^{-1}(t_k)\Lambda_L^{-1}(K), T^i)_{\mathcal{D}}\nabla_{T^k}^R f.
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
w_{B_R}(\kappa(x^i))f &= \{f, B_R^*(\kappa((x^i)'))\}B_R^*(S(\kappa((x^i)''))) = \\
&= -\nabla_{T^i}^R f + (\Lambda_R(K)\kappa(t_k)\Lambda_R^{-1}(K), T^i)_{\mathcal{D}}\nabla_{\kappa(T^k)}^L f.
\end{aligned}$$

Note that $K \in D$ stands for the argument of the functions from $Fun(D)$.

The Lie algebra \mathcal{G} -actions can be lifted to the group G -actions. The corresponding formulae can be written in a compact form by using the maps defined by the global decompositions $D = \kappa(G)B$ and $D = \kappa(B)G$. On the top of the maps $\Lambda_L, \Lambda_R : D \rightarrow B$ we have also the maps $\Xi_L, \Xi_R : D \rightarrow G$ respectively defined as $K = \kappa(\Xi_L(K))\Lambda_R^{-1}(K)$ and $K = \kappa(\Lambda_L(K))\Xi_R^{-1}(K)$, $K \in D$. The actions of \mathcal{G} on D via the vector fields $w_{B_L}(\kappa^{-1}(x^i))$ and $w_{B_R}(\kappa(x^i))$ is then respectively lifted to the G -actions as follows

$$h \triangleright K = \kappa(h)K\Xi_R(\kappa[h\Lambda_L(K)]), \quad h \in G, \quad K \in D, \quad (20a)$$

$$h \triangleright K = \kappa[\Xi_L^{-1}(\Lambda_R^{-1}(K)h^{-1})]Kh^{-1}. \quad h \in G, \quad K \in D. \quad (20b)$$

It is easy to verify that, in both cases, it holds:

$$(h_1 h_2) \triangleright K = h_1 \triangleright (h_2 \triangleright K).$$

In particular, when the cosymmetry group B is Abelian, the G -actions induced by the moment maps B_L and B_R coincide and give nothing but the twisted adjoint action of G on D (i.e. $h \triangleright K = \kappa(h)Kh^{-1}$, $h \in G$, $K \in D$). This fact, that will be proved in Section 4, justifies our terminology "quasi-adjoint" action for the case of non-Abelian cosymmetry groups.

#

3.2 Proper subsymmetry

In the case of the standard Hamiltonian symmetry, every subgroup H of the symmetry group G also realizes the Hamiltonian symmetry. In the general Poisson-Lie context (anomalous or not), such statement is generically false. A natural question then arises: which subgroups of G are themselves Poisson-Lie symmetry groups? We are going to answer this question and we also determine the corresponding moment maps.

Theorem 3: Let D be a decomposable twisted Heisenberg double, κ an automorphism of D preserving B and N a normal subgroup of B . Denote by C the factor group B/N , by ρ the natural homomorphism $B \rightarrow C$ and by $P_\kappa : \text{Lie}(D) \rightarrow \text{Lie}(B)$ a projector on $\text{Lie}(B)$ with kernel $\kappa(\text{Lie}(G))$. Suppose that the Hopf subalgebra $\rho^*(\text{Fun}(C))$ of $\text{Fun}(B)$ is also a Poisson subalgebra. Then it holds: The composed map $\nu_R \equiv \rho \circ \Lambda_R$ realizes Poisson-Lie symmetry of D and the corresponding symmetry group H is the subgroup of G . If, moreover, $P_\kappa(\text{Lie}(H)) \subset \text{Lie}(N)$ then the moment map ν_R is non-anomalous.

Proof: The Poisson-Lie bracket on $\text{Fun}(B)$ naturally induces the Poisson-Lie bracket on $\text{Fun}(C)$ because $\rho^*(\text{Fun}(C))$ is the Poisson subalgebra of $\text{Fun}(B)$. Thus

$$\{\rho^*(u), \rho^*(v)\}_B = \rho^*({u, v}_C), \quad u, v \in \text{Fun}(C).$$

Now define

$$w_{\nu_R}(u)f \equiv \{f, \nu_R^*(u')\}_D \nu_R^*(S_C(u'')), \quad u \in \text{Fun}(C), f \in \text{Fun}(D)$$

and calculate

$$\begin{aligned} w_{\nu_R}({u, v}_C) &= \{f, \nu_R^*({u, v}'_C)\}_D \nu_R^*(S_C({u, v}''_C)) = \\ &= \{f, \Lambda_R^*({\rho^*(u), \rho^*(v)}'_B)\}_D \Lambda_R^*(S_B({\rho^*(u), \rho^*(v)}''_B)) = \\ &= w_{\Lambda_R}({\rho^*(u), \rho^*(v)}_B) = [w_{\Lambda_R}(\rho^*(u)), w_{\Lambda_R}(\rho^*(v))] = [w_{\nu_R}(u), w_{\nu_R}(v)]. \end{aligned}$$

Here we have used the obvious fact that

$$w_{\nu_R}(u) = w_{\Lambda_R}(\rho^*(u)).$$

This fact also directly implies, that H is the subgroup of G .

Let us see how the Lie algebra $Lie(H)$ of H is located in the Lie algebra $Lie(D)$ of the double D . Choose a vector subspace $V \subset Lie(B)$ that is complement to $Lie(N)$ (i.e. $Lie(B) = Lie(N) \dot{+} V$). We can certainly pick a basis $t_i = (t_\iota, t_I)$ such that $t_\iota \in Lie(N)$ and $t_I \in V$ and complete (t_ι, t_I) by the dual basis (T^ι, T^I) of $Lie(G)$. From the duality property (5), it follows that T^ι 's span V^\perp and T^I 's span $Lie(N)^\perp$ (the superscript \perp means "perpendicular" in the sense of the bilinear form $(\cdot, \cdot)_{\mathcal{D}}$). We recall the formula (16b)

$$w_{\Lambda_R}(y)f = -\delta_{t_i}(y)\nabla_{T^i}^R f = -\delta_{t_\iota}(y)\nabla_{T^\iota}^R f - \delta_{t_I}(y)\nabla_{T^I}^R f.$$

If y is in $\rho^*(Fun(C))$, then $\delta_{t_i}(y) = 0$ and we thus obtain

$$w_{\Lambda_R}(y)f = -\delta_{t_I}(y)\nabla_{T^I}^R f.$$

This means that $Lie(H)$ is spanned by T^I 's only, or, in other words, $Lie(H) = Lie(N)^\perp$.

Since the twisting automorphism κ preserves the cosymmetry group B the anomaly matrix $M_{\kappa^{-1}}^{ij}$ (cf. (13)) can be rewritten as

$$M_{\kappa^{-1}}^{ij} = (T^i, \kappa(T^m))_{\mathcal{D}}(\kappa(t_m), T^j)_{\mathcal{D}} = (P_\kappa T^i, T^j)_{\mathcal{D}}. \quad (21)$$

Now we pick $u, v \in Fun(C)$ and, by using (12b) and (21), we calculate

$$\begin{aligned} \{\nu_R^*(u), \nu_R^*(v)\}_D &= \{\Lambda_R^*(\rho^*(u)), \Lambda_R^*(\rho^*(v))\}_D = \\ &= \Lambda_R^*\left(\{\rho^*(u), \rho^*(v)\}_B - M_{\kappa^{-1}}^{ab} \nabla_{t_a}^R \rho^*(u) \nabla_{t_b}^R \rho^*(v)\right) = \\ &= \Lambda_R^*\left(\rho^*(\{u, v\}_C) - (P_\kappa T^A, T^B)_{\mathcal{D}} \nabla_{t_A}^R \rho^*(u) \nabla_{t_B}^R \rho^*(v)\right). \end{aligned}$$

The transition from the second to the third line is justified by the fact that $\nabla_{t_\alpha}^R \rho^*(u) = \nabla_{t_\beta}^R \rho^*(v) = 0$ (Note that $a = (\alpha, A)$, $b = (\beta, B)$.) Since both T^A 's and T^B 's are in $Lie(H) = Lie(N)^\perp$, we have $(P_\kappa T^A, T^B)_{\mathcal{D}} = 0$. Hence we conclude that the moment map ν_R is non-anomalous:

$$\{\nu_R^*(u), \nu_R^*(v)\}_D = \nu_R^*(\{u, v\}_C).$$

#

Remark: We have worked out the subsymmetry story for the right moment map Λ_R . Obviously, there is an analogous "left story" for which the conclusions are the same: a subgroup $H \subset G$ acting from the left (in the κ -twisted way) is the subsymmetry subgroup if $Lie(H) = Lie(N)^\perp$ where $Lie(N)$ is the ideal in the cosymmetry Lie algebra $Lie(B)$. If, moreover, $P_\kappa(Lie(H)) \subset Lie(N)$ then the H -subsymmetry is non-anomalous. We should also remark, that from two conditions $[Lie(B), Lie(N)] \subset Lie(N)$ and $P_\kappa(Lie(H)) \subset Lie(N)$ only the second one is our original result. The first one was already identified in [15, 3] for the non-twisted Heisenberg doubles.

3.3 Improper subsymmetry

In this subsection, we partially release the condition of the decomposability of twisted Heisenberg doubles in the sense that we shall keep the unicity of the decomposition but not the globality. Thus denote O_L the set of elements $K \in D$ for which it exists a $g \in G$ and a $b \in B$ such that $K = \kappa(b)g^{-1}$. In the same way, denote by O_R the set of elements $K \in D$ for which it exists a $\tilde{g} \in G$ and a $\tilde{b} \in B$ such that $K = \kappa(\tilde{g})\tilde{b}^{-1}$. Suppose, moreover, that the respective decompositions $\kappa(B)G$ and $\kappa(G)B$ on O_L and O_R are unique.

In the non-twisted case $\kappa = Id$, it was shown in [1] that the lack of global decomposability has unpleasant consequences. Namely, the fundamental Semenov-Tian-Shansky Poisson structure (6) is no longer symplectic and, therefore, the Poisson manifold $(D, \{.,.\}_D)$ cannot play the role of the phase-space of any dynamical system. It turns out, however, that out from the Poisson structure $\{.,.\}_D$ one can construct symplectic submanifolds of D (called the symplectic leaves) which have the same dimension as D . In particular, Alekseev and Malkin have proved in [1] that the intersection $O_L \cap O_R$ is such symplectic leaf of $(D, \{.,.\}_D)$. The result of Alekseev and Malkin can be generalized to the twisted case as the following Lemma states:

Lemma 3:

Let (D, κ) be a twisted Heisenberg double and M its submanifold defined as $M = O_L \cap O_R$. Consider maps $\Lambda_L : M \rightarrow B$, $\Xi_R : M \rightarrow G$ induced by the unambiguous decomposition $M = \kappa(B)G$ and maps $\Xi_L : M \rightarrow G$, $\Lambda_R : M \rightarrow B$, induced by $M = \kappa(G)B$ (thus $K = \kappa(\Lambda_L(K))\Xi_R^{-1}(K)$ and

$K = \kappa(\Xi_L(K))\Lambda_R^{-1}(K)$ for each K in M). Denote by r_G and r_B the right-invariant Maurer-Cartan forms on G and B , respectively (e.g. if G is a matrix group $r_G = dg g^{-1}$). Then a two-form ω_M on M defined as

$$\omega_M = \frac{1}{2}(\Lambda_L^*(r_B) \wedge \Xi_L^*(r_G))_{\mathcal{D}} + \frac{1}{2}(\Lambda_R^*(r_B) \wedge \Xi_R^*(r_G))_{\mathcal{D}} \quad (22)$$

is symplectic and its inverse is the fundamental Poisson bivector (9) restricted to M .

Proof: Choose a basis t_i of \mathcal{B} and T^i of \mathcal{G} fulfilling the duality relation $(T^i, t_j)_{\mathcal{D}} = \delta_j^i$. The form ω_M can be then rewritten as

$$\omega_M = \frac{1}{2}(\Lambda_L^*(r_B), T^i)_{\mathcal{D}} \wedge (\Xi_L^*(r_G), t_i)_{\mathcal{D}} + \frac{1}{2}(\Lambda_R^*(r_B), T^i)_{\mathcal{D}} \wedge (\Xi_R^*(r_G), t_i)_{\mathcal{D}}.$$

Denote by $\langle ., . \rangle$ the pairing between forms and vectors and recall the definition of the projectors $\Pi_{L\tilde{R}}, \Pi_{\tilde{L}R}, \Pi_{R\tilde{L}}, \Pi_{\tilde{R}L}$ from the proof of the Theorem 1. Then we have

$$\langle (\Lambda_L^*(r_B), T^i)_{\mathcal{D}}, t \rangle = (R_{K*}\kappa(T^i), \Pi_{L\tilde{R}}t)_{\mathcal{D}}, \quad (23a)$$

$$\langle (\Xi_L^*(r_G), t_i)_{\mathcal{D}}, t \rangle = (R_{K*}\kappa(t_i), \Pi_{\tilde{L}R}t)_{\mathcal{D}}, \quad (23b)$$

$$\langle (\Lambda_R^*(r_B), T^i)_{\mathcal{D}}, t \rangle = -(L_{K*}T^i, \Pi_{R\tilde{L}}t)_{\mathcal{D}}, \quad (23c)$$

$$\langle (\Xi_R^*(r_G), t_i)_{\mathcal{D}}, t \rangle = -(L_{K*}t_i, \Pi_{\tilde{R}L}t)_{\mathcal{D}}, \quad (23d)$$

where t is a vector at a point K of $M \subset D$. Let us show how to demonstrate (23abcd) on the example (23a). Due to the decomposability $M = \kappa(B)G$, the vectors $L_{K*}T^i, R_{K*}\kappa(t_i)$ form the basis of the tangent space $T_K M$. Thus it is sufficient to prove (23a) for t being one of the elements of the basis of $T_K M$. For $t = L_{K*}T^j$, it is obvious that the r.h.s. of (23a) vanishes. On the other hand, knowing that $\Lambda_L(Ke^{sT^j}) = \Lambda_L(K)$, we can evaluate the l.h.s.:

$$\langle (\Lambda_L^*(r_B), T^i)_{\mathcal{D}}, L_{K*}T^j \rangle = \langle (r_B, T^i)_{\mathcal{D}}, \Lambda_{L*}(L_{K*}T^j) \rangle = 0.$$

For $t = R_{K*}\kappa(t_j)$, the r.h.s. of (23a) gives

$$(R_{K*}\kappa(T^i), \Pi_{L\tilde{R}}R_{K*}\kappa(t_j))_{\mathcal{D}} = (R_{K*}\kappa(T^i), R_{K*}\kappa(t_j))_{\mathcal{D}} = \delta_j^i.$$

On the other hand, knowing that $\Lambda_L(e^{s\kappa(t_j)}K) = e^{st_j}\Lambda_L(K)$, we can evaluate the l.h.s.:

$$\begin{aligned} & \langle (\Lambda_L^*(r_B), T^i)_{\mathcal{D}}, R_{K*}\kappa(t_j) \rangle = \langle (r_B, T^i)_{\mathcal{D}}, \Lambda_{L*}(R_{K*}\kappa(t_j)) \rangle = \\ & = \langle (r_B, T^i)_{\mathcal{D}}, R_{\Lambda_L(K)*}t_j \rangle = (R_{\Lambda_L^{-1}(K)*}R_{\Lambda_L(K)*}t_j, T^i)_{\mathcal{D}} = (t_j, T^i)_{\mathcal{D}} = \delta_j^i. \end{aligned}$$

By using the relations (23abcd), we can evaluate the form ω_M on any two vectors $t, u \in T_K M$ in terms of the projectors:

$$\begin{aligned} \omega_M(t, u) &= \frac{1}{2}(R_{K*}\kappa(T^i), \Pi_{L\tilde{R}}t)_{\mathcal{D}}(R_{K*}\kappa(t_i), \Pi_{\tilde{L}R}u)_{\mathcal{D}} \\ &\quad - \frac{1}{2}(R_{K*}\kappa(T^i), \Pi_{L\tilde{R}}u)_{\mathcal{D}}(R_{K*}\kappa(t_i), \Pi_{\tilde{L}R}t)_{\mathcal{D}} \\ &\quad + \frac{1}{2}(L_{K*}T^i, \Pi_{R\tilde{L}}t)_{\mathcal{D}}(L_{K*}t_i, \Pi_{\tilde{R}L}u)_{\mathcal{D}} - \frac{1}{2}(L_{K*}T^i, \Pi_{R\tilde{L}}u)_{\mathcal{D}}(L_{K*}t_i, \Pi_{\tilde{R}L}t)_{\mathcal{D}} = \\ &= \frac{1}{2}(\Pi_{L\tilde{R}}t, \Pi_{\tilde{L}R}u)_{\mathcal{D}} - \frac{1}{2}(\Pi_{L\tilde{R}}u, \Pi_{\tilde{L}R}t)_{\mathcal{D}} + \frac{1}{2}(\Pi_{R\tilde{L}}t, \Pi_{\tilde{R}L}u)_{\mathcal{D}} - \frac{1}{2}(\Pi_{R\tilde{L}}u, \Pi_{\tilde{R}L}t)_{\mathcal{D}}. \end{aligned}$$

By realizing that it holds

$$(t, \Pi_{\tilde{L}R}u)_{\mathcal{D}} = (\Pi_{R\tilde{L}}t, \Pi_{\tilde{L}R}u)_{\mathcal{D}} = (\Pi_{R\tilde{L}}t, u)_{\mathcal{D}},$$

$$\Pi_{\tilde{L}R} + \Pi_{R\tilde{L}} = Id,$$

we finally arrive at

$$\omega_M(t, u) = (t, (\Pi_{\tilde{L}R} - \Pi_{L\tilde{R}})u)_{\mathcal{D}}.$$

From the equation (10), we know that the form ω_M is invertible and its inverse is nothing but the Semenov-Tian-Shansky Poisson tensor (9) restricted to M . From this it also follows that ω_M is closed hence symplectic.

#

It is certainly a good news to have the symplectic submanifold M of D , since it allows us to construct dynamical systems also for globally non-decomposable twisted Heisenberg doubles. On the other hand, it is a much less good news to remark that nothing guarantees that the group G still acts on M . In fact, it turns out, generically, that the submanifold M of D is not invariant under the left or right action of G on D , therefore G cannot play the role of the symmetry group. It may happen, however, that there is

a subgroup H of G which does preserve the submanifold M and which has the property that $\mathcal{H} = \mathcal{N}^\perp$, where \mathcal{N} is an ideal in \mathcal{B} . We have then the following lemma

Lemma 4: Let H be a subgroup of G preserving the submanifold $M = O_L \cap O_R$. We suppose moreover that $\mathcal{H} = \mathcal{N}^\perp$, where \mathcal{N} is the ideal of \mathcal{B} . Then there exists a moment map $\nu : M \rightarrow B$ realizing the global (H, C) -Poisson-Lie symmetry of M .

Proof: For concreteness, we speak about the right action of G on D . Sitting on M , we construct the map $w_{\Lambda_R} : Fun(B) \rightarrow Vect(M)$ by using the formula (3):

$$w_{\Lambda_R}(y)f = \{f, \Lambda_R^*(y')\}_M \Lambda_R^*(S(y'')), \quad y \in Fun(B), f \in Fun(M).$$

For every $y \in Fun(B)$, we have obviously

$$\nabla_{\kappa(T^i)}^L \Lambda_R^*(y) = 0.$$

Since the Poisson bivector on M is given by Eq.(9), we thus obtain

$$w_{\Lambda_R}(y)f = \nabla_{T^i}^R f \nabla_{t_i}^R \Lambda_R^*(y') \Lambda_R^*(S(y'')) = -\nabla_{T^i}^R f \Lambda_R^*((\nabla_{t_i}^L y') S(y'')) = -\delta_{t_i}(y) \nabla_{T^i}^R f.$$

It follows that the Lie algebra \mathcal{G} of G does act M , however, because we have supposed it, this action cannot be lifted to the action of G itself. Similarly as in the demonstration of Theorem 3, we thus observe that for $\nu_R \equiv \rho \circ \Lambda_R$ the following is true

$$\{f, \nu_R^*(u')\}_M \nu_R^*(S(u'')) = -\delta_{t_i}(\rho^*(u)) \nabla_{T^i}^R f, \quad u \in Fun(C), f \in Fun(M).$$

Recall that T^I 's span the Lie algebra $\mathcal{H} = \mathcal{N}^\perp$ therefore ν_R is indeed the moment map realizing the action of \mathcal{H} on M . This action can be obviously lifted to the action of the group H on M , since we have supposed that M is H -invariant.

#

Remark:

In the case of the non-decomposable Heisenberg doubles of the type just described we cannot speak about the proper subsymmetry since G does not act on M , therefore we speak about the improper subsymmetry.

Now it is time for an example. Consider a group $SL(3, R)$ (consisting of real 3×3 -matrices of unit determinant) and denote by $sl(3, R)$ its Lie algebra (consisting of real traceless 3×3 -matrices). The direct product $D = sl(3, R) \times SL(3, R)$ can be equipped with the group structure as follows:

$$(\chi, g)(\tilde{\chi}, \tilde{g}) = (\chi + Ad_g \tilde{\chi}, g\tilde{g}), \quad \chi, \tilde{\chi} \in sl(3, R), \quad g, \tilde{g} \in SL(3, R),$$

$$(\chi, g)^{-1} = (-Ad_{g^{-1}} \chi, g^{-1}).$$

The Lie algebra \mathcal{D} of D is formed by pairs of elements of $sl(3, R)$ written as $\phi \oplus \alpha$ with the commutator

$$[\phi \oplus \alpha, \psi \oplus \beta] = ([\phi, \psi] + [\alpha, \psi]) \oplus [\alpha, \beta].$$

There is a natural bi-invariant metric on D induced from an invariant bilinear form $(\cdot, \cdot)_{\mathcal{D}}$ on $\mathcal{D} = Lie(D)$:

$$(\phi \oplus \alpha, \psi \oplus \beta)_{\mathcal{D}} = Tr(\phi\beta) + Tr(\psi\alpha), \quad \alpha, \beta, \phi, \psi \in sl(3, R).$$

The twisting automorphism κ is defined by

$$\kappa(\chi, g) = (-\chi^T, (g^{-1})^T),$$

where T stands for matrix transposition. In order to establish that (D, κ) is indeed a twisted Heisenberg double, we have to identify two maximally isotropic subgroups. Here they are

$$G = \{(\chi, g) \in D; \chi = 0\},$$

$$B = \left\{ (\chi, g) \in D; \chi = \begin{pmatrix} \chi^{\triangleleft} + \chi^{\triangleright} & \chi^{1+} & \chi^{3+} \\ \chi^{1-} & -2\chi^{\triangleright} & \chi^{2+} \\ \frac{1}{\varepsilon}(1 - e^{-\varepsilon s}) & \chi^{2-} & -\chi^{\triangleleft} + \chi^{\triangleright} \end{pmatrix}, g = \begin{pmatrix} e^{\frac{1}{2}\varepsilon s} & 0 & -\varepsilon e^{\frac{1}{2}\varepsilon s} \chi^{\triangleleft} \\ 0 & 1 & 0 \\ 0 & 0 & e^{-\frac{1}{2}\varepsilon s} \end{pmatrix} \right\},$$

where $s, \chi^{\triangleleft}, \chi^{\triangleright}, \chi^{j+}, \chi^{j-}, \chi^{2-} \in \mathbf{R}$ are coordinates on B and ε is a parameter.

For the basis of \mathcal{D} , we may choose

$$T^{\triangleleft} = 0 \oplus H, \quad T^{\triangleright} = 0 \oplus \frac{K}{3}, \quad t_{\triangleleft} = 2H \oplus (-\varepsilon E^{3+}), \quad t_{\triangleright} = 2K \oplus 0,$$

$$T^{j+} = 0 \oplus E^{j+}, \quad T^{j-} = 0 \oplus E^{j-}, \quad t_{j+} = E^{j-} \oplus 0, \quad t_{j-} = E^{j+} \oplus 0, \quad j = 1, 2,$$

$$T^{3+} = 0 \oplus E^{3+}, \quad T^{3-} = 0 \oplus E^{3-}, \quad t_{3+} = E^{3-} \oplus \varepsilon H, \quad t_{3-} = E^{3+} \oplus 0,$$

where

$$\begin{aligned} E^{1+} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & E^{2+} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, & E^{3+} &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ E^{1-} &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & E^{2-} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, & E^{3-} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\ H &= \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}, & K &= \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}. \end{aligned}$$

It is easy to verify that it holds

$$(t_i, t_j)_{\mathcal{D}} = 0, \quad (T^i, T^j)_{\mathcal{D}} = 0, \quad (T^i, t_j)_{\mathcal{D}} = \delta_j^i, \quad i, j = \triangleleft, \triangleright, 1\pm, 2\pm, 3\pm.$$

The commutation relations of $\mathcal{G} = \text{Span}(T^i)$ are evidently those of the Lie algebra $sl(3, R)$. It is important for us to give the complete list of (non-zero) commutators of $\mathcal{B} = \text{Span}(t_i)$. Thus we have

$$[t_{\triangleleft}, t_{1+}] = \varepsilon t_{2-}, \quad [t_{\triangleleft}, t_{2+}] = -\varepsilon t_{1-}, \quad [t_{3+}, t_{3-}] = \varepsilon t_{3-}, \quad [t_{3+}, t_{\triangleleft}] = \varepsilon t_{\triangleleft},$$

$$[t_{3+}, t_{j\pm}] = \mp \frac{1}{2} \varepsilon t_{j\pm}, \quad j = 1, 2$$

Let us choose a (nilpotent) subalgebra \mathcal{H} of $\mathcal{G} = sl(3, R)$ spanned by T^{j+} . Thus the only non-zero commutator is

$$[T^{1+}, T^{2+}] = T^{3+}.$$

It is easy to find $\mathcal{N} \subset \mathcal{B}$ such that $\mathcal{H} = \mathcal{N}^\perp$: we have

$$\mathcal{N} = \text{Span}(t_{\triangleleft}, t_{\triangleright}, t_{j-}), \quad j = 1, 2, 3.$$

It is the matter of direct check to verify that \mathcal{N} is indeed an ideal in \mathcal{B} . Therefore the (Heisenberg) group H consisting of upper-triangular real matrices with units on the diagonal is a good candidate for the Poisson-Lie subsymmetry. The corresponding cosymmetry group C has Lie algebra $\mathcal{C} = \mathcal{B}/\mathcal{N}$ and,

by slightly abusing the notation, we can denote its basis by t_{j+} , $j = 1, 2, 3$. The non-zero commutators of \mathcal{C} read

$$[t_{3+}, t_{j+}] = -\frac{1}{2}\varepsilon t_{j+}, \quad j = 1, 2.$$

The cosymmetry group C can be most easily described in the dual way. Denote the coordinate functions as ξ^j , $j = 1, 2, 3$. The coproduct reads

$$\begin{aligned} \Delta \xi^3 &= \xi^3 \otimes 1 + 1 \otimes \xi^3, \\ \Delta \xi_j &= \xi_j \otimes 1 + e^{-\frac{\varepsilon}{2}\xi^3} \otimes \xi_j, \end{aligned}$$

the antipode

$$S(\xi_3) = -\xi_3, \quad S(\xi_j) = -e^{\frac{\varepsilon}{2}\xi^3}\xi_j, \quad j = 1, 2$$

and the counit

$$\epsilon(\xi_j) = 0, \quad j = 1, 2, 3.$$

The dual map $\rho^* : Fun(C) \rightarrow Fun(B)$ reads

$$\rho^*(\xi_3) = s, \quad \rho^*(\xi_j) = \chi^{j-}, \quad j = 1, 2.$$

The Poisson-Lie bracket on $Fun(C)$ comes from that on $Fun(B)$, which, in turn, is given by (11). The result of the computation reads

$$\{\xi^1, \xi^2\}_C = \frac{1}{\varepsilon}(1 - e^{-\varepsilon\xi^3}), \quad \{\xi^3, \xi^j\}_C = 0, \quad j = 1, 2.$$

We observe that both symmetry group H and the cosymmetry group C are non-Abelian.

Let us now show that the (H, C) -Poisson-Lie subsymmetry is in fact improper. In order to see this, we first notice that the Heisenberg double D is non-decomposable since e.g. the element

$$(\chi, g) = \left(\begin{pmatrix} 0 & -\frac{1}{\varepsilon} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

cannot be written as $\kappa(b)g^{-1}$ for some $b \in B$ and $g \in G$.

It is easy to identify the manifold $M = O_L \cap O_R$. We find

$$M = \left\{ (\chi, g) \in D; \operatorname{Tr}(J_L E^{3-}) > -\frac{1}{\varepsilon}, \operatorname{Tr}(J_R E^{3+}) < \frac{1}{\varepsilon} \right\}, \quad (24)$$

where we have defined the $sl(3, R)$ -valued functions J_L, J_R on D as

$$J_L(\chi, g) = \chi, \quad J_R(\chi, g) = -\operatorname{Ad}_{g^{-1}} \chi.$$

The symplectic form on M can be computed from the explicit expression (22). The result of calculation is as follows

$$\begin{aligned} \omega_M = & -\frac{1}{2} \operatorname{Tr}(dJ_R \wedge l_G) + \frac{1}{2} \operatorname{Tr}(dJ_L \wedge r_G) + \\ & -\frac{\varepsilon}{2} \frac{\operatorname{Tr}(dJ_L H) \wedge \operatorname{Tr}(dJ_L E^{3-})}{1 + \varepsilon \operatorname{Tr}(J_L E^{3-})} - \frac{\varepsilon}{2} \frac{\operatorname{Tr}(dJ_R H) \wedge \operatorname{Tr}(dJ_R E^{3+})}{1 - \varepsilon \operatorname{Tr}(J_R E^{3+})}. \end{aligned}$$

Note that the left and right-invariant Maurer-Cartan forms l_G, r_G can be written also as $g^{-1}dg, dgg^{-1}$ since $G = SL(3, R)$ is the matrix group. The explicit expression of the symplectic form ω_M is quite illuminating in the sense that it explains why the constraints $\operatorname{Tr}(J_L E^{3-}) > -\frac{1}{\varepsilon}, \operatorname{Tr}(J_R E^{3+}) < \frac{1}{\varepsilon}$ in (24) had to be imposed. It is now the matter of direct inspection to find that the right action of the group H on D and the left action of $\kappa(H)$ on D preserve, respectively, the symplectic manifold $M = O_L \cap O_R$. The (H, C) -Poisson-Lie symmetry of (M, ω_M) is therefore established.

4 u -deformed WZW model and its gauging

We begin this section by introducing a particular example of the deformation of the WZW model which was not discussed in [9, 10, 11]. Then we shall perform the symplectic reduction of this u -deformed WZW model with respect to a non-anomalous quasi-adjoint action submoment map which is a sort of combination of the moment maps constructed in Secs. 3.1 and 3.2. Finally, we shall argue why this quasi-adjoint symplectic reduction can be interpreted as the gauging of the deformed WZW model.

4.1 The u -deformation of the WZW model

It was conjectured in [9] and explained in detail in [11] that the standard WZW model [17] on a compact Lie group K is a dynamical system whose phase space can be identified with certain (decomposable) twisted Heisenberg double of a loop group LK . Moreover, the symplectic form of the WZW model is just the inverse of the fundamental Semenov-Tian-Shansky Poisson bivector (9). The basic idea of the article [9] can be rephrased as follows: since the loop group LK may possess several different twisted Heisenberg doubles (D, κ) , it makes sense to consider the dynamical system based on each of (D, κ) as a sort of generalized WZW model. The (twisted Heisenberg) double of the standard WZW model is distinguished among all other doubles of the loop group LK by the fact that the cosymmetry group B is Abelian. This circumstance is reflected by the fact that the standard WZW model has the ordinary Hamiltonian symmetry structure. On the other hand, the generalized WZW models have necessarily non-Abelian cosymmetry groups therefore their symmetry structure must be genuinely Poisson-Lie. Some generalized WZW models form naturally families parametrized by one or several parameters. Suppose we investigate such a family. If for a particular value of the parameters the corresponding generalized WZW model becomes the standard WZW model, we call the other members of this family the deformed WZW models.

Let us now describe a particular family of the deformed WZW models, which was not discussed in [9, 10, 11]. Thus K be a connected simple compact Lie group whose Lie algebra \mathcal{K} is equipped with a non-degenerate Ad -invariant bilinear form $(\cdot, \cdot)_{\mathcal{K}}$. Let LK be the group of smooth maps from a circle S^1 into K (the group law is given by pointwise multiplication) and define a natural non-degenerate Ad -invariant bilinear form $(\cdot | \cdot)$ on $L\mathcal{K} \equiv Lie(LK)$ by the following formula

$$(\alpha | \beta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\sigma (\alpha(\sigma), \beta(\sigma))_{\mathcal{K}}, \quad (25)$$

As the twisted Heisenberg double D , we take the semidirect product of the loop group LK with its Lie algebra $L\mathcal{K}$. Thus the group multiplication law on D reads

$$(\chi, g) \cdot (\tilde{\chi}, \tilde{g}) = (\chi + Ad_g \tilde{\chi}, g\tilde{g}), \quad g \in LK, \chi \in L\mathcal{K}, \quad (26a)$$

$$(\chi, g)^{-1} = (-Ad_{g^{-1}}\chi, g^{-1}), \quad (26b)$$

and the Lie algebra \mathcal{D} of D has the structure of semidirect sum $\mathcal{D} = L\mathcal{K} \overset{\leftarrow}{\oplus} L\mathcal{K}$

$$[\phi \oplus \alpha, \psi \oplus \beta] = ([\phi, \psi] + [\alpha, \psi], [\alpha, \beta]).$$

Here $\phi, \psi \in L\mathcal{K}$ are in the first and $\alpha, \beta \in L\mathcal{K}$ in the second component of the semidirect sum. The bi-invariant metric on D comes from Ad -invariant bilinear form $(\cdot, \cdot)_{\mathcal{D}}$ on $Lie(D) = \mathcal{D}$ defined with the help of (25):

$$(\phi \oplus \alpha, \psi \oplus \beta)_{\mathcal{D}} = (\phi|\beta) + (\psi|\alpha).$$

The metric preserving automorphism κ of the group D reads

$$\kappa(\chi, g) = (\chi + k\partial_{\sigma}gg^{-1}, g), \quad (26c)$$

where k is an (integer) parameter. The maximally isotropic subgroups are

$$G = \{(\chi, g) \in D; \chi = 0\}, \quad (27a)$$

$$B = \{(\chi, g) \in D; g = e^{u(\chi)}\}, \quad (27b)$$

where u is a certain map from $L\mathcal{K}$ to the Cartan subalgebra \mathcal{T} of $L\mathcal{K}$. Let us now explain the construction of the map u : The group K is naturally embedded in LK as the subgroup consisting of constant loops. The maximal torus T of K is therefore the (Abelian) subgroup of LK and we call $\mathcal{T} = Lie(T)$ the Cartan subalgebra of $L\mathcal{K}$. Since we have the inner product (25) on $L\mathcal{K}$ we can define the orthogonal projector $\mathcal{P}_0 : L\mathcal{K} \rightarrow \mathcal{T}$. Let $U : \mathcal{T} \rightarrow \mathcal{T}$ be a skew-symmetric linear operator, i.e. it holds

$$(Ua, b)_{\mathcal{K}} = -(a, Ub)_{\mathcal{K}}, \quad a, b \in \mathcal{T}. \quad (28)$$

We then define

$$u = U \circ \mathcal{P}_0.$$

It is easy to see that

$$u(\chi) + u(\tilde{\chi}) = u\left(\chi + e^{u(\chi)}\tilde{\chi}e^{-u(\chi)}\right), \quad \chi, \tilde{\chi} \in L\mathcal{K},$$

hence the set B defined by (27b) is indeed the subgroup of D . Moreover, the condition (28) implies the isotropy of B in D .

It is a simple task to establish the decompositions $D = \kappa(G)B$ and $D = \kappa(B)G$. Indeed, we have for every $g \in LK, \chi \in LK$

$$(\chi, g) = (k\partial_\sigma g g^{-1}, g e^{u(J_R)}) (-e^{-u(J_R)} J_R e^{u(J_R)}, e^{-u(J_R)}) = (J_L, e^{u(J_L)}) \cdot (0, e^{-u(J_L)} g),$$

where LK -valued functions J_L, J_R on D are defined as

$$J_L(\chi, g) \equiv \chi, \quad J_R(\chi, g) = -Ad_{g^{-1}}\chi + kg^{-1}\partial_\sigma g. \quad (29a)$$

Thus we can identify the moment maps $\Lambda_{L,R} : D \rightarrow B, \Xi_{L,R} : D \rightarrow G$:

$$\Lambda_L(\chi, g) = (J_L, e^{u(J_L)}), \quad \Lambda_R(\chi, g) = (J_R, e^{u(J_R)}), \quad (29b)$$

$$\Xi_L(\chi, g) = g e^{u(J_R)}, \quad \Xi_R(\chi, g) = g^{-1} e^{u(J_L)}.$$

Now we use the formula (22) and write down the symplectic form ω_u of the u -deformed WZW model:

$$\omega_u = \frac{1}{2}(dJ_L \wedge |r_{LK}) - \frac{1}{2}(dJ_R \wedge |l_{LK}) + \frac{1}{2}(u(dJ_L) \wedge |dJ_L) + \frac{1}{2}(u(dJ_R) \wedge |dJ_R). \quad (30)$$

Here $r_{LK} = dg g^{-1}$ and $l_{LK} = g^{-1}dg$ stand for the right and the left-invariant Maurer-Cartan forms on the group manifold LK .

The role of the deformation parameter is played by the linear operator U . Indeed, if $U \rightarrow 0$ the form ω_u can be rewritten as

$$\omega_{u=0} = d(J_L |r_{LK}) + \frac{1}{2}k(r_{LG} \wedge |\partial_\sigma r_{LG}).$$

In the expression $\omega_{u=0}$, we can recognize the symplectic form of the standard WZW model (cf. [9, 5, 2]). We now complete the definition of the u -deformed WZW model by saying that it is a dynamical system with the phase space D , with the symplectic form ω_u and with the following Hamiltonian

$$H = -\frac{1}{2k}(J_L |J_L) - \frac{1}{2k}(J_R |J_R). \quad (31)$$

We note without giving proof that, in distinction to the q -deformation of the WZW model introduced in [9], the u -deformation does preserve the conformal symmetry.

Let us study the symmetry structure of the u -WZW model. The group $G = LK$ acts from the left as

$$h \triangleright (\chi, g) = \kappa((0, h)).(\chi, g) = (k\partial_\sigma h h^{-1} + h\chi h^{-1}, hg), \quad h, g \in LK, \quad \chi \in LK$$

and also from the right

$$(\chi, g) \triangleleft h = (\chi, g)(0, h^{-1}) = (\chi, gh^{-1}).$$

We know (by construction) that both these actions are Poisson-Lie symmetries with the moment maps $\Lambda_{L,R}$ given by (29b). Now we are going to evaluate the (anomalous) Poisson brackets (12ab) of the moment maps. First of all, we have to describe the structure of the cosymmetry group B in the dual language. The complexified algebra $Fun^{\mathbf{C}}(B)$ is generated by (linear) functions $F^{\alpha,n}, F^{\mu,n}$ defined as

$$F^{\alpha,n}(\chi) = (E^{\alpha,n}|\chi), \quad F^{\mu,n}(\chi) = (H^{\mu,n}|\chi), \quad \chi \in LK. \quad (32)$$

Here $E^{\alpha,n} = E^\alpha e^{in\sigma}$ and E^α are the step generators of the complexified Lie algebra $K^{\mathbf{C}}$. On the other hand, $H^{\mu,n} = H^\mu e^{in\sigma}$ where H^μ are the (orthonormalized) Cartan generators fulfilling the relations

$$\begin{aligned} [H^\mu, E^\alpha] &= \langle \alpha, H^\mu \rangle E^\alpha, \quad [E^\alpha, E^{-\alpha}] = \alpha^\vee, \quad [E^\alpha, E^\beta] = c^{\alpha\beta} E^{\alpha+\beta}, \\ (H^\mu, H^\nu)_K &= \delta^{\mu\nu}, \quad (E^\alpha, E^{-\alpha})_{K^{\mathbf{C}}} = \frac{2}{|\alpha|^2}, \quad (E^\alpha)^\dagger = E^{-\alpha}, \quad (H^\mu)^\dagger = H^\mu, \end{aligned}$$

where the coroot α^\vee is defined as

$$\alpha^\vee = \frac{2}{|\alpha|^2} \langle \alpha, H^\mu \rangle H^\mu.$$

Obviously, $E^{\alpha,n}, H^{\mu,n}$, $n \in \mathbf{Z}$ is the basis of $LK^{\mathbf{C}}$. The (non-Abelian) group law on B is encoded in the coproduct, the antipode and the counit on $Fun^{\mathbf{C}}(B)$. From the Eqs. (26), (27b) and (32), it is not difficult to find out:

$$\begin{aligned} \Delta F^{\mu,n} &= F^{\mu,n} \otimes 1 + 1 \otimes F^{\mu,n}, \quad S(F^{\mu,n}) = -F^{\mu,n}, \quad \varepsilon(F^{\mu,n}) = 0, \quad \varepsilon(F^{\alpha,n}) = 0, \\ \Delta F^{\alpha,n} &= F^{\alpha,n} \otimes 1 + e^{-\langle \alpha, U(H^\mu) \rangle F^{\mu,0}} \otimes F^{\alpha,n}, \quad S(F^{\alpha,n}) = -e^{\langle \alpha, U(H^\mu) \rangle F^{\mu,0}} F^{\alpha,n}. \end{aligned}$$

Because of the fact that $\chi^\dagger = -\chi$, the operation of the complex conjugation \dagger on $Fun^{\mathbf{C}}(B)$ is given by

$$(F^{\alpha,n})^\dagger = -F^{-\alpha,-n}, \quad (F^{\mu,n})^\dagger = -F^{\mu,-n}.$$

It can be then easily verified that

$$\Delta \circ \dagger = (\dagger \otimes \dagger) \circ \Delta, \quad S \circ \dagger = \dagger \circ S, \quad \varepsilon \circ \dagger = \dagger \circ \varepsilon.$$

This means that Δ, S, ε descend from $Fun^{\mathbf{C}}(B)$ to $Fun^{\mathbf{R}}(B)$ making the latter the real commutative Hopf algebra dual to the real group B .

The Poisson-Lie bracket on $Fun^{\mathbf{C}}(B)$ can be obtained from the general formula (11):

$$\begin{aligned} \{F^{\mu,m}, F^{\nu,n}\}_B &= 0, \\ \{F^{\mu,m}, F^{\alpha,n}\}_B &= \langle \alpha, H^\mu \rangle F^{\alpha,m+n}, \\ \{F^{\alpha,m}, F^{-\alpha,n}\}_B &= \frac{2}{|\alpha|^2} \langle \alpha, H^\mu \rangle F^{\mu,m+n}, \\ \{F^{\alpha,m}, F^{\beta,n}\}_B &= c^{\alpha\beta} F^{\alpha+\beta,m+n} - \langle \alpha, U(H^\mu) \rangle \langle \beta, H^\mu \rangle F^{\alpha,m} F^{\beta,n}. \end{aligned}$$

It is easy to verify, that the Poisson-Lie bracket on $Fun^{\mathbf{C}}(B)$ verifies

$$\{f_1^\dagger, f_2^\dagger\}_B = \{f_1, f_2\}_B^\dagger,$$

hence it defines also the Poisson-Lie bracket on the real group B . Now we are ready to evaluate the anomalous Poisson brackets (12ab). We start with

$$\begin{aligned} \Lambda_L^*(F^{\alpha,n}) &= (J_L | E^\alpha e^{in\sigma}) \equiv J_L^{\alpha,n}, \quad \Lambda_L^*(F^{\mu,n}) = (J_L | H^\mu e^{in\sigma}) \equiv J_L^{\mu,n}, \\ \Lambda_R^*(F^{\alpha,n}) &= (J_R | E^\alpha e^{in\sigma}) \equiv J_R^{\alpha,n}, \quad \Lambda_R^*(F^{\mu,n}) = (J_R | H^\mu e^{in\sigma}) \equiv J_R^{\mu,n} \end{aligned}$$

and find

$$\begin{aligned} \{J_L^{\mu,m}, J_L^{\nu,n}\}_D &= k\delta^{\mu\nu} in\delta_{m+n,0}, \\ \{J_L^{\mu,m}, J_L^{\alpha,n}\}_D &= \langle \alpha, H^\mu \rangle J_L^{\alpha,n+m}, \\ \{J_L^{\alpha,m}, J_L^{-\alpha,n}\}_D &= \frac{2}{|\alpha|^2} \left(\langle \alpha, H^\mu \rangle J_L^{\mu,n+m} + ikn\delta_{m+n,0} \right), \end{aligned}$$

$$\{J_L^{\alpha,m}, J_L^{\beta,n}\}_D = c^{\alpha\beta} J_L^{\alpha+\beta,m+n} - \underline{< \alpha, U(H^\mu) > < \beta, H^\mu > J_L^{\alpha,m} J_L^{\beta,n}}; \quad (33a)$$

$$\begin{aligned} \{J_R^{\mu,m}, J_R^{\nu,n}\}_D &= -k\delta^{\mu\nu} in\delta_{m+n,0}, \\ \{J_R^{\mu,m}, J_R^{\alpha,n}\}_D &= < \alpha, H^\mu > J_R^{\alpha,n+m}, \\ \{J_R^{\alpha,m}, J_R^{-\alpha,n}\}_D &= \frac{2}{|\alpha|^2} \left(< \alpha, H^\mu > J_R^{\mu,n+m} - ikn\delta_{m+n,0} \right), \\ \{J_R^{\alpha,m}, J_R^{\beta,n}\}_D &= c^{\alpha\beta} J_R^{\alpha+\beta,m+n} - \underline{< \alpha, U(H^\mu) > < \beta, H^\mu > J_R^{\alpha,m} J_R^{\beta,n}}; \end{aligned} \quad (33b)$$

$$\{J_L, J_R\}_D = 0. \quad (33c)$$

In the formulae above, we note the anomalous terms proportional to k . They correspond to the matrices M_κ^{ij} and $M_{\kappa-1}^{ij}$ in (12a) and (12b), respectively. We remark, that the left and right brackets differ by the sign in front of k . This fact will be crucial for gauging the u -deformed WZW model in Sec 4.3. We have also underlined the deformation terms containing U . Thus the relations (33a) or (33b) can be referred to as those of u -deformed Kac-Moody algebra.

Knowing the symplectic structure of the u -deformed WZW models, we can compute other interesting Poisson brackets. The observables on D are functions of $\chi \in L\mathcal{K}$ and $g \in LK$. Let us consider two functions $\phi(g), \psi(g)$, which do not depend on χ . Then we find directly from (6):

$$\{\phi(g), \psi(g)\}_D = \underline{\nabla_{T^\mu}^R \phi(g) \nabla_{U(T^\mu)}^R \psi(g) - \nabla_{U(T^\mu)}^L \phi(g) \nabla_{T^\mu}^L \psi(g)},$$

where $T^\mu \equiv iH^\mu \in \mathcal{T} \subset \mathcal{K}$. Note, that we have again underlined the u -deformation term (the corresponding bracket of the standard WZW model vanishes). Finally, we have

$$\begin{aligned} \{\phi(g), J_L^{\mu,m}\}_D &= \nabla_{H^{\mu,m}}^L \phi(g), \\ \{\phi(g), J_L^{\alpha,n}\}_D &= \underline{\nabla_{E^{\alpha,n}}^L \phi(g) - i < \alpha, U(H^\mu) > J_L^{\alpha,n} \nabla_{T^\mu}^L \phi(g)}, \\ \{\phi(g), J_R^{\mu,m}\}_D &= -\nabla_{H^{\mu,m}}^R \phi(g), \\ \{\phi(g), J_R^{\alpha,n}\}_D &= \underline{-\nabla_{E^{\alpha,n}}^R \phi(g) + i < \alpha, U(H^\mu) > J_R^{\alpha,n} \nabla_{T^\mu}^R \phi(g)}, \end{aligned}$$

4.2 Symplectic reduction: generalities

The symplectic reduction is the method of construction of new symplectic manifolds out from old ones. The simplest way of explaining the method relies on the dual language which uses rather the algebra of functions $Fun(M)$ on a symplectic manifold M than the manifold M itself. We note that the space $Fun(M)$ is the Poisson algebra, i.e. the Lie algebra compatible with the structure of the (standard commutative point-wise) multiplication on $Fun(M)$. The Lie commutator is nothing but the Poisson bracket $\{.,.\}_M$ corresponding to a symplectic structure ω_M on M and the compatibility condition is given by the Leibniz rule:

$$\{f, gh\}_M = \{f, g\}_M h + \{f, h\}_M g, \quad f, g, h \in Fun(M).$$

Let J be an ideal of the algebra $Fun(M)$ with respect to the ordinary commutative multiplication on $Fun(M)$ (typically, J is the ideal of functions vanishing on a submanifold $N \subset M$). Let J be also the Poisson subalgebra of $Fun(M)$, i.e. $\{J, J\} \subset J$. We can now construct a new Poisson algebra \tilde{A} defined as follows

$$\tilde{A} = \{f \in Fun(M); \{f, J\}_M \in J\}.$$

Note that the property $\{J, J\} \subset J$ implies that $J \subset \tilde{A}$. By construction, J is not only the ordinary ideal of \tilde{A} but it is also the Poisson ideal, i.e. $\{\tilde{A}, J\}_M \subset J$. Obviously, the factor algebra $A_r \equiv \tilde{A}/J$ inherits the Poisson bracket from \tilde{A} hence it becomes itself the Poisson algebra. If J is the ideal of functions vanishing on a submanifold $N \subset M$, then the algebra A_r is nothing but the Poisson algebra of functions corresponding to some symplectic manifold M_r . The manifold M_r together with its corresponding Poisson bracket $\{.,.\}_r$ (or, equivalently, with its symplectic form ω_r) is called the reduced symplectic manifold. If there is a Hamiltonian H on M such that $H \in \tilde{A}$, its class in \tilde{A}/J is denoted as H_r and it is referred to as the reduced Hamiltonian.

The symplectic reduction is often put in relation with the actions of Lie groups on the non-reduced manifold M . It may even happen that the reader used to the group approach to the symplectic reduction did not recognize at first reading that his way of thinking about the reduction is just a particular

case of the general algebraic definition presented above. We believe that it is worth to elucidate this point not only for pedagogical reasons. In fact, the group-based symplectic reduction will turn out to be in the core of our gauging of the u -WZW model. We shall work in the general Poisson-Lie setting, the standard Hamiltonian symplectic reduction (cf. [14] and references therein) will be the special case of our discussion when the cosymmetry group B is Abelian.

Suppose that there is a non-anomalous moment map $\mu : M \rightarrow B$ realizing the (G, B) -Poisson-Lie symmetry of M (cf. the Definition 1 of Section 2.2). Due to the property (2b) of the Poisson-Lie bracket on $Fun(B)$, we know that the kernel of the counit $Ker(\epsilon)$ is the Poisson subalgebra of $(Fun(B), \{.,.\}_B)$. Since the moment map μ is non-anomalous, the pull-back $\mu^*(Ker(\epsilon))$ is also the Poisson subalgebra of $(Fun(M), \{.,.\}_M)$. Thus the role of the ideal J from the general definition above is played by the ideal of $Fun(M)$ generated by $\mu^*(Ker(\epsilon))$. We denote it also by the letter J . In the situation just described, the resulting reduced symplectic manifold M_r (corresponding to the reduced Poisson algebra \tilde{A}/J), can be easily "visualised". For this, let us suppose that the set P of points of M mapped by μ to the unit element e of the cosymmetry group B forms a smooth submanifold of M . It is not difficult to verify that the action of the symmetry group G (which is itself locally induced by the moment map μ) leaves P invariant. Let us moreover suppose that the G -action on P is free, or, in other words, that P is isomorphic to a principal G -bundle. Then the basis P/G of this G -fibration can be then identified with the reduced symplectic manifold M_r . The restriction of the symplectic form ω on P becomes degenerated and the degeneracy direction of ω turn out to be nothing but the orbits of the gauge group G . Thus the symplectic form ω_r is naturally induced from ω . Indeed, on each local trivialisation of the G -bundle P we can choose a slice. The restriction of ω on the slice is the reduced symplectic form ω_r .

A particularly good situation occurs when the G fibration of P is topologically trivial. In this case, one can visualize the reduced symplectic manifold as the submanifold of P (and, hence, as the submanifold of the original symplectic manifold M). This can be done by choosing a global slice $Q_i = 0$, where the functions Q_i are in $Fun(M)$. In the usual terminology, the functions $J_i \in \mu^*(Ker(\epsilon)) \subset Fun(M)$ are called the first class constraints and

the functions Q_i their complementary second class constraints. The reduced symplectic manifold M_r is now the common locus of all constraints $J_i = 0$ and $Q_i = 0$ and the reduced symplectic form ω_r is the pull-back of the non-reduced form ω to the submanifold M_r .

It is sometimes convenient to fix the gauge only partially. This means that it exists a slice $Q_\gamma = 0$ (the subscript γ runs over a smaller set than the subscript i) which restricts the gauge freedom to some subgroup $H \subset G$. If we note by the letter L the common locus $J_i = 0, Q_\gamma = 0$ in M , the reduced symplectic manifold M_r can be identified with the coset space L/H . The interest in such partial gauge fixing will be evident in the studies of the symplectic structure of the standard gauged WZW model and of its deformations. Indeed, as we shall see in the following section, there exists the partial gauge fixing for which the manifold L has a very simple left-right chiral symmetric description and the residual gauge group H is finite dimensional, compact and Abelian.

4.3 Symplectic reduction of the u -WZW model

We start this section by remarking that the twisting automorphism κ given by (26c) not only preserves the cosymmetry group B described in (27b) but it leaves invariant every element of B . This means that we can safely apply the Theorem 2 of Sec. 3.1. which now states that the products $\Lambda_L \Lambda_R \equiv B_L$ and $\Lambda_R \Lambda_L \equiv B_R$ are both non-anomalous moment maps. We already know from the general theory that both B_L and B_R realize the global Poisson-Lie symmetries of the twisted Heisenberg double (D, κ) therefore, via their corresponding maps w_{B_L}, w_{B_R} (cf. (3)), they induce the respective actions (20a), (20b) of the loop group $G = LK$ on (D, κ) .

Let us work, for concreteness, with the moment map $B_L = \Lambda_L \Lambda_R$. Recall the group multiplication law in B :

$$(\chi_1, e^{u(\chi_1)}) \cdot (\chi_2, e^{u(\chi_2)}) = (\chi_1 + e^{u(\chi_1)} \chi_2 e^{-u(\chi_1)}, e^{u(\chi_1) + u(\chi_2)}), \quad \chi_1, \chi_2 \in LK. \quad (34)$$

The formula (34) together with Eqs. (29b) allow us to calculate the $B_{L,R}^*$ -pull-backs of the basic functions from $Fun^C(B)$:

$$\begin{aligned} B_L^*(F^{\alpha,n}) &= (\Lambda_L \Lambda_R)^*(F^{\alpha,n}) = J_L^{\alpha,n} + e^{-\langle \alpha, U(H^\mu) J_L^{\mu,0} \rangle} J_R^{\alpha,n}, \\ B_R^*(F^{\alpha,n}) &= (\Lambda_R \Lambda_L)^*(F^{\alpha,n}) = J_R^{\alpha,n} + e^{-\langle \alpha, U(H^\mu) J_R^{\mu,0} \rangle} J_L^{\alpha,n}, \end{aligned}$$

$$B_L^*(F^{\mu,n}) = B_R^*(F^{\mu,n}) = J_L^{\mu,n} + J_R^{\mu,n}.$$

Now we are ready to make explicit the map $w_{B_L} : Fun(B) \rightarrow Vect(D)$:

$$\begin{aligned} w_{B_L}(F^{\alpha,n})f &\equiv \{f, B_L^*((F^{\alpha,n})')\}_D B_L^*(S((F^{\alpha,n})'')) = \\ &= \nabla_{\kappa(E^{\alpha,n})}^L f - e^{-\langle \alpha, U(H^\mu) \rangle} J_L^{\mu,0} \nabla_{E^{\alpha,n}}^R f - \langle \alpha, U(H^\mu) \rangle J_L^{\alpha,n} \nabla_{H^\mu}^R f, \\ w_{B_L}(F^{\mu,n})f &\equiv \{f, B_L^*((F^{\mu,n})')\}_D B_L^*(S((F^{\mu,n})'')) = \\ &= \nabla_{\kappa(H^{\mu,n})}^L f - \nabla_{H^{\mu,n}}^R f, \quad f \in Fun^C(D). \end{aligned}$$

Recall that the symbol $w_{B_L}(F^{\alpha,n})$ denotes the (complex) vector field on D corresponding to the Poisson-Lie Hamiltonian $F^{\alpha,n} \in Fun^C(B)$. Similarly, we find

$$\begin{aligned} w_{B_R}(F^{\alpha,n})f &\equiv \{f, B_R^*((F^{\alpha,n})')\}_D B_R^*(S((F^{\alpha,n})'')) = \\ &= -\nabla_{E^{\alpha,n}}^R f + e^{-\langle \alpha, U(H^\mu) \rangle} J_R^{\mu,0} \nabla_{\kappa(E^{\alpha,n})}^L f + \langle \alpha, U(H^\mu) \rangle J_R^{\alpha,n} \nabla_{H^\mu}^L f, \\ w_{B_R}(F^{\mu,n})f &\equiv \{f, B_R^*((F^{\mu,n})')\}_D B_R^*(S((F^{\mu,n})'')) = \\ &= \nabla_{\kappa(H^{\mu,n})}^L f - \nabla_{H^{\mu,n}}^R f, \quad f \in Fun^C(D). \end{aligned}$$

It is the matter of easy check that the vector fields $w_{B_L}(F^{\alpha,n}), w_{B_L}(F^{\mu,n})$ and also $w_{B_R}(F^{\alpha,n}), w_{B_R}(F^{\mu,n})$ generate the actions of the Lie algebra LK^C on $Fun^C(D)$. Moreover, it can be also seen that, by considering only the Poisson-Lie Hamiltonians from $Fun^R(B)$, these actions get restricted to the actions of LK on $Fun^R(D)$. It is not difficult to lift the LK actions just described to the LK actions. The resulting formulae are the special cases of the general formulae (20a) and (20b):

$$h \triangleright (\chi, g) = \kappa(h)(\chi, g)h_L^{-1}, \quad h_L = e^{-u(hJ_L h^{-1} + \kappa \partial h h^{-1})} h e^{u(J_L)}, \quad h \in LK, \quad (35a)$$

$$h \triangleright (\chi, g) = \kappa(h_R)(\chi, g)h^{-1}, \quad h_R = e^{-u(hJ_R h^{-1} - \kappa \partial h h^{-1})} h e^{u(J_R)}, \quad h \in LK. \quad (35b)$$

We notice that for $U \rightarrow 0$ the cosymmetry group B becomes Abelian and the LK -actions (35a) and (35b) coincide and (as we have promised to show in Section 3.1) they become identical to the twisted adjoint action $h \triangleright (\chi, g) = \kappa(h)(\chi, g)h^{-1}$.

Let Υ be a subset of the set of all positive roots of the Lie algebra $\mathcal{K}^{\mathbf{C}}$. Consider a complex vector space $\mathcal{S}^{\mathbf{C}}$ defined as

$$\mathcal{S}^{\mathbf{C}} = \text{Span}\{E^{\gamma}, E^{-\gamma}, [E^{\gamma}, E^{-\gamma}]\}, \quad \gamma \in \Upsilon.$$

In the rest of this paper, we shall suppose that the subset Υ was chosen in such a way that the vector space $\mathcal{S}^{\mathbf{C}}$ is the Lie subalgebra of $\mathcal{K}^{\mathbf{C}}$ (as an example take the block diagonal embedding of sl_3 in sl_4). Obviously, the vector space

$$\mathcal{T}_S^{\mathbf{C}} = \text{Span}\{[E^{\gamma}, E^{-\gamma}]\}, \quad \gamma \in \Upsilon$$

is the Cartan subalgebra of $\mathcal{S}^{\mathbf{C}}$. The complex Lie algebra $\mathcal{S}^{\mathbf{C}}$ has a natural compact real form \mathcal{S} consisting of the anti-Hermitian elements of $\mathcal{S}^{\mathbf{C}}$. Consider the corresponding compact semi-simple group S and view it as the subgroup of K . We are now going to establish the conditions on the operator U which will guarantee that the action of the loop group LS on D via (35a) or (35b) is the Poisson-Lie subsymmetry.

Suppose that for all $\gamma \in \Upsilon$, the operator $U : \mathcal{T} \rightarrow \mathcal{T}$ fulfils the following condition

$$(\gamma \circ U)(\mathcal{T}_S^{\perp}) = 0, \tag{36}$$

where the subscript \perp stands for the orthogonal complement with respect to the restriction of the Killing-Cartan form $(\cdot, \cdot)_{\mathcal{K}}$ to \mathcal{T} . It is then easy to verify that the set

$$N = \{(\chi, g) \in D; g = e^{u(\chi)}, \chi \in \mathcal{S}^{\perp}\}$$

is the normal subgroup of B . Consider the algebra of complex functions on the group $C = B/N$. As we have learned in Section 3.2, $\text{Fun}^{\mathbf{C}}(C)$ can be injected by the map ρ^* into $\text{Fun}^{\mathbf{C}}(B)$. (Note that ρ^* is the dual map to the projection homomorphism $\rho : B \rightarrow B/C$.) It is easy to see that $\rho^*(\text{Fun}^{\mathbf{C}}(C))$ is spanned by the functions $F^{\gamma, n}, F^{\nu, n}$ where $\gamma \in \Upsilon$ and $H^{\nu} \in \mathcal{T}_S$. The normality of the subgroup N implies that the vector space $\rho^*(\text{Fun}^{\mathbf{C}}(C))$ is in fact the Hopf subalgebra of $\text{Fun}^{\mathbf{C}}(B)$. By using the explicit form of the Poisson-Lie brackets on $\text{Fun}^{\mathbf{C}}(B)$, it is straightforward to check that $\rho^*(\text{Fun}^{\mathbf{C}}(C))$ is also the Poisson subalgebra of $\text{Fun}^{\mathbf{C}}(B)$. It is moreover true that $\rho^*(\text{Fun}^{\mathbf{C}}(C))$ is \dagger -invariant hence we conclude that $\rho^*(\text{Fun}(C))$ is the Poisson subalgebra of $\text{Fun}(B)$. All that means that we can use the Theorem

3 of Section 3.2 to conclude that the action of the loop group LS on D via (35ab) is the Poisson-Lie subsymmetry. Our next goal is to gauge this (non-anomalous) subsymmetry, or, in other words, to perform the symplectic reduction with respect to it.

Consider the LS -subsymmetry moment map $C_L = \rho \circ B_L$, where ρ is the projection homomorphism from B to $C = B/N$. The first step of the reduction procedure consists in identification of the submanifold $P_L \subset D$ such that every point $p \in P_L$ is mapped by C_L to the unit element of the group C . It is easy to see that

$$P_L = \{p \in D; J_L^{\gamma,n}(p) + e^{-\langle \gamma, U(H^\nu) \rangle J_L^{\nu,0}(p)} J_R^{\gamma,n}(p) = 0, J_L^{\nu,n}(p) + J_L^{\nu,n}(p) = 0\},$$

where $\gamma \in \pm \Upsilon$ and ν is such that $H^\nu \in \mathcal{T}_S$. In physicists' terminology, the expressions

$$J_L^{\gamma,n} + e^{-\langle \gamma, U(H^\nu) \rangle J_L^{\nu,0}} J_R^{\gamma,n} = 0, J_L^{\nu,n} + J_L^{\nu,n} = 0 \quad (37)$$

are the first class constraints since it is not difficult to verify that the Poisson brackets of the constraints among themselves as well as those of the Hamiltonian (31) with the constraints vanish on the constrained surface P_L .

Now the u -deformed WZW symplectic form ω_u restricted to P_L becomes degenerated in the directions of the action of LS on P_L . As we already know from Section 4.2, the reduced symplectic manifold M_r can be identified with the coset space P_L/LS . We now perform a partial gauge fixing (cf. the general discussion in Section 4.2) which will lead to very elegant left-right symmetric chiral description of the symplectic structure of the reduced symplectic manifold M_r . For this, we first study the action of LS on D given by the formula (35a). By using the formula (7a), we rewrite it as follows

$$s \triangleright (\chi, g) = (s\chi s^{-1} + k\partial_\sigma s s^{-1}, s g s_L^{-1}), \quad s_L = e^{-u(sJ_L s^{-1} + \kappa \partial s s^{-1})} s e^{u(J_L)}, \quad s \in LK. \quad (38)$$

It is convenient to decompose χ as $\chi_s + \chi_p$, where $\chi_s \in L\mathcal{S}$ and $\chi_p \in L\mathcal{S}^\perp$. We thus see from Eq. (38) that χ_s and χ_p do not mix under the action of s . We know that every χ_s can be brought by some s to an element of the finite dimensional Cartan subalgebra \mathcal{T}_S (cf. [9], Theorem 3.6). Having in mind the definition (29a) of J_L , this leads to the following natural slice on D :

$$J_L^{\gamma,n} = 0, \quad \gamma \in \pm \Upsilon, \quad n \in \mathbf{Z}, \quad (39a)$$

$$J_L^{\nu,n} = 0, \quad n \in \mathbf{Z}, \quad n \neq 0, \quad (39b)$$

where ν is such that $H^\nu \in \mathcal{T}_S$. This slice is partial (it corresponds to the slice $Q_\gamma = 0$ in the general discussion of Sec. 4.2). Indeed, the residual gauge group H is the normalizer of the Cartan subalgebra \mathcal{T}_S and, as the discussion before the Theorem 3.6 of [9] implies, the finite-dimensional Cartan torus T_S is the normal subgroup of H . (In fact H/T_S is nothing but the affine Weyl group of LS). The constraints (37) and (39) can be now rewritten in a U -independent way as

$$J_L^{\gamma,n} = 0, \quad J_R^{\gamma,n} = 0, \quad \gamma \in \pm\Upsilon, \quad n \in \mathbf{Z}, \quad (40a)$$

$$J_L^{\nu,n} = 0, \quad J_R^{\nu,n} = 0, \quad n \in \mathbf{Z}, \quad n \neq 0. \quad (40b)$$

$$J_L^{\nu,0} + J_R^{\nu,0} = 0, \quad (40c)$$

where ν is such that $H^\nu \in \mathcal{T}_S$. The constraints (40) define the submanifold $L \subset D$ and the reduced symplectic manifold M_r can be identified with the space of cosets L/H .

The similar discussion can be performed also with the moment map $C_R = \rho \circ B_R$. The first class constrained manifold P_R is

$$P_R = \{p \in D; \quad J_R^{\gamma,n}(p) + e^{-\langle \gamma, U(H^\nu) \rangle J_R^{\nu,0}(p)} J_L^{\gamma,n}(p) = 0, \quad J_L^{\nu,n}(p) + J_R^{\nu,n}(p) = 0\}, \quad (41)$$

where $n \in \mathbf{Z}$, $\gamma \in \pm\Upsilon$ and ν is such that $H^\nu \in \mathcal{T}_S$. The partial slice on D is

$$J_R^{\gamma,n} = 0, \quad \gamma \in \pm\Upsilon, \quad n \in \mathbf{Z}, \quad (42a)$$

$$J_R^{\nu,n} = 0, \quad n \in \mathbf{Z}, \quad n \neq 0, \quad (42b)$$

where ν is such that $H^\nu \in \mathcal{T}_S$. The constraints (41) and (42) can also be rewritten in the U -independent way as

$$J_L^{\gamma,n} = 0, \quad J_R^{\gamma,n} = 0, \quad \gamma \in \pm\Upsilon, \quad n \in \mathbf{Z}, \quad (43a)$$

$$J_L^{\nu,n} = 0, \quad J_R^{\nu,n} = 0, \quad n \in \mathbf{Z}, \quad n \neq 0. \quad (43b)$$

$$J_L^{\nu,0} + J_R^{\nu,0} = 0, \quad (43c)$$

We thus see that the symplectic reduction based on the moment map B_R gives the same result as the one based on B_L . This happens inspite of the

fact that w_{C_L} and w_{C_R} induce the different actions of the gauge group LS on D .

Our next task will be the description of the symplectic form ω_r on M_r . Actually, we shall describe the pull-back of the original Semenov-Tian-Shansky form ω_u on D to the submanifold $L \subset D$. We again use the Theorem 3.6 of [9] which permits us to parametrize the Heisenberg double D by means of two elements g_L, g_R of LK and one element μ of the Weyl alcove \mathcal{A}_K in the Cartan subalgebra $\mathcal{T}_K \subset \mathcal{K}$:

$$(\chi, g) = \kappa(0, g_L)(\mu, e_{LK})(0, g_R)^{-1} = (g_L \mu g_L^{-1} + k \partial_\sigma g_L g_L^{-1}, g_L g_R^{-1}). \quad (44)$$

Here e_{LK} is the unit element in LK . The Semenov-Tian-Shansky form ω_u given by (30) gets rewritten in the new variables as follows

$$\begin{aligned} \tilde{\omega}_u = & -d(\mu | g_R^{-1} dg_R) + \frac{k}{2}(g_R^{-1} dg_R \wedge |\partial(g_R^{-1} dg_R)) + \frac{1}{2}(u(dJ_R) \wedge |dJ_R) + \\ & + d(\mu | g_L^{-1} dg_L) - \frac{k}{2}(g_L^{-1} dg_L \wedge |\partial(g_L^{-1} dg_L)) + \frac{1}{2}(u(dJ_L) \wedge |dJ_L), \end{aligned} \quad (45)$$

where

$$\begin{aligned} J_L &= g_L \mu g_L^{-1} + k \partial_\sigma g_L g_L^{-1}, \\ J_R &= -g_R \mu g_R^{-1} - k \partial_\sigma g_R g_R^{-1}. \end{aligned}$$

Before giving the interpretation of the reduced symplectic manifold in terms of the deformed gauged WZW model, let us first study the residual gauge symmetries of the form $\tilde{\omega}_u$. We recall that the residual gauge group H is the normalizer of the Cartan algebra \mathcal{T}_S . We can make it smaller by further gauge fixing. Thus we suppose that the variable $J_L^{\nu,0} (= -J_R^{\nu,0})$ takes values only in the Weyl alcove of \mathcal{T}_S . (We remind that the Weyl alcove is the fundamental domain of the action of the affine Weyl group of LS on \mathcal{T}_S). With this restriction the residual gauge group becomes just the Cartan torus T_S acting as

$$t_S \triangleright (g_L, g_R) = (t_S g_L, t_S g_R), \quad t_S \in T_S. \quad (46)$$

Indeed, replacing $g_{L,R}$ by $t_S g_{L,R}$ in (45), the form $\tilde{\omega}_u$ transforms as

$$\tilde{\omega}_u \rightarrow \tilde{\omega}_u + d(J_L + J_R | t_S^{-1} dt_S) = \tilde{\omega}_u,$$

since the term $d(J_L + J_R|t_S^{-1}dt_S)$ vanishes due to the constraint $J_L^{\nu,0} + J_R^{\nu,0} = 0$. It is important to stress that the parametrization (44) of the double D via the variables μ, g_L, g_R gave rise to another gauge symmetry of the form $\tilde{\omega}_u$ which is related to the ambiguity of the chiral decomposition (44). Indeed, if we pick arbitrary element t_K from the Cartan torus T_K then it holds

$$(\chi, g) = \kappa(0, g_L)(\mu, e_{LK})(0, g_R)^{-1} = \kappa(0, g_L t_K)(\mu, e_{LK})(0, g_R t_K)^{-1}.$$

This means that the full residual gauge group of the form $\tilde{\omega}_u$ is $T_S \times T_K$ acting as

$$(t_S, t_K) \triangleright (g_L, g_R) = (t_S g_L t_K, t_S g_R t_K), \quad t_S \in T_S, t_K \in T_K.$$

The reader may find strange that we have somewhat artificially augmented the residual gauge symmetry of the Semenov-Tian-Shansky form ω_u by expressing it in the new ambiguous variables μ, g_L, g_R . However, the benefit of this parametrization consists in the fact that in the form $\tilde{\omega}_u$ the variables g_L and g_R get disentangled. The form $\tilde{\omega}_u$ is defined on the manifold $LK \times \mathcal{A}_K \times LK$ and its pull-back on D via the map (44) gives the Semenov-Tian-Shansky form ω_u . Obviously, it holds $D = (LK \times \mathcal{A}_K \times LK)/T_K$. We conclude this section by an observation, that the Hamiltonian (31) of the u -WZW model descends to the reduced Hamiltonian H_r (cf. the general discussion in Section 4.2). Thus our symplectic reduction has produced a new dynamical system (M_r, ω_r, H_r) that will be interpreted in the next subsection as the deformed gauged WZW model.

4.4 Interpretation

The gauged WZW model is a dynamical system and its symplectic structure has been thoroughly investigated e.g. in Sec. 3.2 and in Appendix A of [7]. We report here Gawędzki's results in the language of the left-right movers, by considering maps $m_L, m_R : \mathbf{R} \rightarrow K$ fulfilling

$$(\partial_\xi m_{L,R} m_{L,R}^{-1}, \mathcal{S})_K = 0, \quad (47a)$$

$$m_{L,R}(\xi + 2\pi) = e^{-\frac{2\pi\nu}{k}} m_{L,R}(\xi) e^{\frac{2\pi\mu}{k}}, \quad (47b)$$

where μ is in the Weyl alcove of \mathcal{T}_K and ν in the Weyl alcove of \mathcal{T}_S . The symplectic form of the gauged WZW model is then given by the following

expression (cf. Eq. (A.1) of [7])

$$\begin{aligned}\omega^{K/S} = & -\frac{k}{2}(m_L^{-1}dm_L \wedge |\partial_\xi(m_L^{-1}dm_L)) + \frac{k}{2}(m_R^{-1}dm_R \wedge |\partial_\xi(m_R^{-1}dm_R)) \\ & -\frac{1}{2}((m_L^{-1}dm_L)(0) - m_L(0)^{-1}\frac{2\pi d\nu}{k}m_L(0), \wedge d\mu)_\mathcal{K} - \frac{1}{2}((dm_L m_L^{-1})(0), \wedge d\nu)_\mathcal{K} \\ & +\frac{1}{2}((m_R^{-1}dm_R)(0) - m_R(0)^{-1}\frac{2\pi d\nu}{k}m_R(0), \wedge d\mu)_\mathcal{K} + \frac{1}{2}((dm_R m_R^{-1})(0), \wedge d\nu)_\mathcal{K}.\end{aligned}$$

In writing the form $\omega^{K/S}$, we have switched from Gawędzki's notations to ours (e.g. we have used $(\cdot, \cdot)_\mathcal{K}$ instead of $\text{Tr}(\cdot, \cdot)$ etc.), nevertheless $\omega^{K/S}$ still does not quite resemble our reduced form $\tilde{\omega}_{u=0}$. In fact, we should note that Gawędzki's chiral movers are quasiperiodic (cf. (47b)) while we use the periodic fields $g_{L,R}(\sigma)$. Indeed, if we perform a transformation

$$m_{L,R}(\xi) = e^{-\frac{\nu\xi}{k}} g_{L,R}(\xi) e^{\frac{\mu\xi}{k}},$$

the conditions (47) become

$$(g_{L,R}\mu g_{L,R}^{-1} + k\partial_\sigma g_{L,R}g_{L,R}^{-1} - \nu, \mathcal{S})_\mathcal{K} = 0, \quad (48a)$$

$$g_{L,R}(\xi + 2\pi) = g_{L,R}(\xi) \quad (48b)$$

and the form $\omega^{K/S}$ transforms to

$$\omega^{K/S} = d(\mu|g_L^{-1}dg_L - g_R^{-1}dg_R) - \frac{k}{2}(g_L^{-1}dg_L \wedge |\partial(g_L^{-1}dg_L)) + \frac{k}{2}(g_R^{-1}dg_R \wedge |\partial(g_R^{-1}dg_R)). \quad (49)$$

It is not difficult to find out that the form (49) coincides with the form $\tilde{\omega}_{u=0}$ given by (45) and the constraints (48a) are, respectively, the constraints (40).

We observe that the symplectic reduction of the u -WZW model for $U = 0$ gives the standard gauged WZW model. Therefore, if we switch on a non-trivial U , we interpret the reduced theory as the u -deformed gauged WZW model.

5 Conclusions and outlook

In the present paper, we have presented a thorough discussion of the gauging of the deformed WZW models. After the general derivation of the quasi-adjoint actions (20a) and (20b), which are to be gauged in general case, we have worked out the u -deformed WZW model as an example. Moreover, in Sections 3.2 and 3.3, we have also introduced the moment maps $\rho \circ \Lambda_{L,R}$ which can be used for deforming the procedure of the null gauging of the WZW models [4, 13] .

The main open issue concerning the deformed WZW models is a quantization. Since we dispose of the rather explicit description of the Poisson brackets of the deformed WZW models (cf. Section 4.1) it seems to be doable to identify the operator algebra of the quantum deformed model and also the unitary representations of this algebra. What seems to be more difficult, however, is to extract from the deformed WZW theories general axioms of the deformed vertex algebras. We find this problem exciting and we wish to deal with it in future.

References

- [1] A.Yu. Alekseev and A.Z. Malkin, Symplectic structures associated to Lie-Poisson groups, *Commun. Math. Phys.* **162** (1994) 147-174, [hep-th/9303038]
- [2] J. Balog, L. Fehér and L. Palla, Chiral Extensions of the WZNW Phase Space, Poisson-Lie Symmetries and Groupoids, *Nucl.Phys.* **B568** (2000) 503-542 , [hep-th/9910046]
- [3] H. Flaschka and T. Ratiu, Convexity theorem for Poisson actions of compact Lie groups, *Ann. Sci. Ecole Norm. Sup.* **29** (1996) 787-809
- [4] P. Forgács, A. Wipf, J. Balog, L. Fehér and L. O’Raifeartaigh, Liouville and Toda theories as conformally reduced WZNW theories, *Phys. Lett.* **B227** (1989) 214-220
- [5] K. Gawędzki, Classical origin of quantum group symmetries in WZW conformal field theory, *Commun. Math. Phys.* **139** (1991) 201-213

- [6] K. Gawędzki, Topological actions in two-dimensional quantum field theories, In: *Non-perturbative quantum field theory*, eds. *G. t Hooft, A. Jaffe, G. Mack, P. K. Mitter, R. Stora* (Plenum Press, New York, 1988) p.101-141
- [7] K. Gawędzki, Boundary WZW, G/H, G/G and CS theories, *Annales Henri Poincaré* **3** (2002) 847-881, [hep-th/0108044]
- [8] C. Kassel, *Quantum Groups*, Springer -Verlag 1995
- [9] C. Klimčík, Quasitriangular WZW model, *Rev. Math. Phys.* **16** (2004) 679-808, [hep-th/0103118]
- [10] C. Klimčík, Quasitriangular chiral WZW model in a nutshell, *Prog. Theor. Phys.Suppl* **144** (2001) 119-124, [hep-th/0108148]
- [11] C. Klimčík, Poisson-Lie symmetry and q -WZW model, to appear in the *Proceedings of the 4th International Symposium On Quantum Theory And Symmetries (QTS-4), Varna Free University, Bulgaria, 15-21 August 2005*, [hep-th/0511003]
- [12] C. Klimčík and P. Ševera, Open strings and D-branes in WZNW model, *Nucl. Phys.* **B488** (1997) 653-676, [hep-th/9609112]
- [13] C. Klimčík and A.A. Tseytlin, Exact four-dimensional string solutions and Toda-like sigma models from 'null-gauged' WZNW theories, *Nucl. Phys.* **B424** (1994) 71-96, [hep-th/9402120]
- [14] J-P. Ortega, T. Ratiu, *Momentum maps and Hamiltonian reduction* (Birkhauser, Boston, 2004)
- [15] M. Semenov-Tian-Shansky, Dressing transformations and Poisson groups actions, *Publ.RIMS* **21**, Kyoto Univ. (1985) 1237-1260
- [16] M. Semenov-Tian-Shansky, Poisson Lie Groups, Quantum Duality Principle and the Twisted Quantum Double, *Theor. Math. Phys.* **93** (1992) 1292-1307, [hep-th/9304042]
- [17] E. Witten, Non-Abelian bosonisation in two dimensions, *Commun. Math. Phys.* **92** (1984) 455-472

- [18] E. Witten, On holomorphic factorization of WZW and coset models,
Commun. Math. Phys. **144** (1992) 189-212